Term functions in multialgebra theory

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Multialgebra

An *n-ary multioperation* f on a set A is a mapping

$$f : A^n \to P^*(A).$$

($P^*(A)$ denotes the set of the nonempty subsets of A).

Let $\mathcal{F}$ be a type of (multi)algebras.

A *multialgebra* $\mathbf{A} = (A, F)$ of type $\mathcal{F}$ consists of a set A and a family of multioperations F obtained by associating a multioperation $f^{\mathbf{A}}$ (or, simply, f) on A to each symbol f from $\mathcal{F}$.

The tools

- *The universal algebra of the nonempty subsets of a multialgebra* (Pickett): each multialgebra **A** determines a universal algebra $\mathbf{P}^*(\mathbf{A})$ on $P^*(A)$ defining

$$f^{\mathbf{P}^*(\mathbf{A})}(A_1,\dots,A_n)=\bigcup\{f^{\mathbf{A}}(a_1,\dots,a_n)\mid\ a_i\in A_i,\ i=1,\dots,n\}$$

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- the clone $\mathrm{Clo}(\mathbf{P}^*(\mathbf{A}))$ of the term functions of $\mathbf{P}^*(\mathbf{A})$
- *Identities:* if $q, \ r$ are n-ary terms of type $\mathcal{F}$,

$$q = r \text{ on } \mathbf{A} \Leftrightarrow q^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_n) = r^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_n), \quad \forall a_1, \ldots, a_n \in A,$$

$$q \cap r \neq \emptyset \text{ on } \mathbf{A} \Leftrightarrow q^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_n) \cap r^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_n) \neq \emptyset, \quad \forall a_1, \ldots, a_n \in A.$$

Submultialgebras

Let $\mathbf{A} = (A, F)$ be a multialgebra.
A subset $B \subseteq A$ is a *submultialgebra* of $\mathbf{A}$ if for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$ and any $b_1, \ldots, b_n \in B$

$$f^{\mathbf{A}}(b_1, \ldots, b_n) \subseteq B.$$

Theorem (Pickett)

A subset $B \subseteq A$ is a submultialgebra of $\mathbf{A}$ if and only if $P^*(B)$ is a subalgebra of $\mathbf{P}^*(\mathbf{A})$.

$\Rightarrow$ If B is a submultialgebra of $\mathbf{A}$, t is an n-ary term of type $\mathcal{F}$ and $b_1, \ldots, b_n \in B$ then

$$t^{\mathbf{P}^*(\mathbf{A})}(b_1, \ldots, b_n) \subseteq B.$$

$\Rightarrow$ (Madarasz) If $X \subseteq A$ then $a \in [X]$ if and only if there exist $n \in \mathbb{N}$, an n-ary term t and $x_1, \ldots, x_n \in X$ such that

$$a \in t^{\mathbf{P}^*(\mathbf{A})}(x_1, \ldots, x_n).$$

Homomorphisms

Let $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ be multialgebras of type $\mathcal{F}$ and $h : A \to B$.

- h is a *multialgebra homomorphism* from $\mathbf{A}$ into $\mathbf{B}$ if for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$ and any $a_1, \ldots, a_n \in A$,

$$h(f^{\mathbf{A}}(a_1, \ldots, a_n)) \subseteq f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

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- h is an *ideal homomorphism* if for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$ and any $a_1, \ldots, a_n \in A$,

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- the *multialgebra isomorphisms* are the bijective ideal homomorphisms.

Homomorphisms

The mapping h induces the mapping

$$h_* : P^*(A) \to P^*(B),\ \ h_*(X) = h(X) = \{h(x) \mid x \in X\}.$$

Theorem (Pickett)

h is an ideal multialgebra homomorphism if and only if h_* is a universal algebra homomorphism.

$\Rightarrow$ (Madarasz) If $h : A \to B$ is an ideal homomorphism, t is an n-ary term of type $\mathcal{F}$ and $a_1, \ldots, a_n \in A$ then

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Factor multialgebras

- Let $\mathbf{A} = (A, F)$ be a multialgebra of type $\mathcal{F}$ and let ρ be an equivalence relation on A. Defining for each $n \in \mathbb{N}$ and each $f \in \mathcal{F}_n$,

  $$f(a_1/\rho, \ldots, a_n/\rho) = \{b/\rho \mid b \in f(b_1, \ldots, b_n),\ a_i \rho b_i,\ i = 1, \ldots, n\}$$

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  Representation Theorem (Grätzer)

  Any multialgebra is a factor of a universal algebra modulo an appropriate equivalence relation.

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  $\{b/\rho | b \in t^{\mathbf{P}^*(\mathbf{A})}(b_1, ..., b_n), a_i \rho b_i, i = \overline{1, n}\} \subseteq t^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, ..., a_n/\rho).$

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- Even if **A** is a universal algebra, the above inclusion is not, in general, an equality, therefore the identities of **A** become, in general, only weak identities of $\mathbf{A}/\rho$.

Example

If $(\mathbb{Z}_5, +)$ is the cyclic group of order 5 and we take the equivalence

$$\rho = \{0,1\} \times \{0,1\} \cup \{2\} \times \{2\} \cup \{3,4\} \times \{3,4\},$$

$(\mathbb{Z}_5/\rho, +)$ is a multialgebra with a binary multioperation $+$ given by

$+$	$\{0,1\}$	$\{2\}$	$\{3,4\}$
$\{0,1\}$	$\{0,1\},\{2\}$	$\{2\},\{3,4\}$	$\{0,1\},\{3,4\}$
$\{2\}$	$\{2\},\{3,4\}$	$\{3,4\}$	$\{0,1\}$
$\{3,4\}$	$\{0,1\},\{3,4\}$	$\{0,1\}$	$\{0,1\},\{2\},\{3,4\}$

$$\{c/\rho \mid c = (b_0 + b_1) + b_2, b_0 = b_1 = 2, b_2 \in \{3,4\}\} = \{2/\rho, 3/\rho\}$$

$$(2/\rho + 2/\rho) + 3/\rho = 3/\rho + 3/\rho = \{0/\rho, 2/\rho, 3/\rho\},$$

and the associativity is only weakly satisfied on $(\mathbb{Z}_5/\rho, +)$ since

$$2/\rho + (2/\rho + 3/\rho) = 2/\rho + 0/\rho = \{2/\rho, 3/\rho\}.$$

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- Example (Roth)

  Let $(G, \cdot)$ be a finite group and ρ the conjugacy relation on G. Then $(G/\rho, \cdot, 1/\rho)$ is a multialgebra with a binary multioperation and a nullary operation, and both multialgebras $(G, \cdot, 1)$ and $(G/\rho, \cdot, 1/\rho)$ satisfy the (strong) identities:

$$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3),\ x_1 \cdot 1 = 1 \cdot x_1 = x_1.$$

Ideal equivalences

Let $\mathbf{A} = (A, F)$ be a multialgebra. An equivalence relation ρ on A determines on $P^*(A)$ the relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ defined by:

$$X\overline{\rho}Y \Leftrightarrow \forall x \in X,\ \exists y \in Y:\ x\rho y \text{ and } \forall y \in Y,\ \exists x \in X:\ x\rho y;$$

$$X\overline{\overline{\rho}}Y \ \Leftrightarrow\ x\rho y,\ \forall x \in X,\ \forall y \in Y \ \Leftrightarrow\ X \times Y \subseteq \rho.$$

The relation ρ is an *ideal equivalence* on $\mathbf{A}$ if for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$ and any $x_i, y_i \in A$ for which $x_i \rho y_i$ $(i \in \{1, \ldots, n\})$ we have

$$a \in f^{\mathbf{A}}(x_1, \ldots, x_n) \Rightarrow \exists b \in f^{\mathbf{A}}(y_1, \ldots, y_n):\ a\rho b.$$

Ideal equivalences

Theorem (Pickett)

If ρ is an ideal equivalence on $\mathbf{A}$ then the canonical projection

$$\pi_\rho : A \to A/\rho,\ \pi_\rho(a) = a/\rho$$

is an ideal homomorphism. Conversely, if $h : A \to B$ is an ideal homomorphism from $\mathbf{A}$ into $\mathbf{B}$, its kernel is an ideal equivalence on $\mathbf{A}$. Moreover, the mapping $h(a) \mapsto \pi_{\ker h}(a)$ is an multialgebra isomorphism between $h(\mathbf{A})$ and $\mathbf{A}/\ker h$.

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$\Rightarrow$ the equivalence relations ρ for which the factor multialgebra $\mathbf{A}/\rho$ is a universal algebra are particular ideal equivalence relations

- (Breaz, Pelea) An equivalence relation ρ of the multialgebra $\mathbf{A}$ is ideal if and only if $\overline{\rho}$ is a congruence relation on $\mathbf{P}^*(\mathbf{A})$.

Corollary

Let $\mathbf{A} = (A, F)$ *be a multialgebra and let* ρ *be an equivalence relation on* A*. The following conditions are equivalent:*

(a) ρ *is an ideal equivalence on* $\mathbf{A}$*;*

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(b) *for any* $n \in \mathbb{N}$, *any* $f \in \mathcal{F}_n$, *and any* $x_i, y_i \in A$ *for which* $x_i \rho y_i$ $(i \in \{1, \ldots, n\})$ *we have*

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$$f^{\mathbf{A}}(x_1, \dots, x_n) \overline{\rho} f^{\mathbf{A}}(y_1, \dots, y_n);$$

(c) *for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$, any $a, b, x_1, \dots, x_n \in A$ such that $a \rho b$, and any $i \in \{1, \dots, n\}$, we have*

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(d) *for any n-ary term t of type $\mathcal{F}$, and any $x_i, y_i \in A$ with $x_i \rho y_i$ ($i \in \{1, \ldots, n\}$) we have*

$$t^{\mathbf{P}^*(\mathbf{A})}(x_1, \ldots, x_n) \, \overline{\rho} \, t^{\mathbf{P}^*(\mathbf{A})}(y_1, \ldots, y_n).$$

Direct products of multialgebras

If $(\mathbf{A}_i \mid i \in I)$ is a family of multialgebras of type $\mathcal{F}$, the Cartesian product $\prod_{i\in I} A_i$ with the multioperations defined by

$$f^{\prod_{i\in I}\mathbf{A}_i}(a_1,\ldots,a_n) = \prod_{i\in I} f^{\mathbf{A}_i}(a_1(i),\ldots,a_n(i))$$

is a multialgebra called *the direct product of the multialgebras* $(\mathbf{A}_i \mid i \in I)$.

- If t is an n-ary term of type $\mathcal{F}$ and $a_1,\ldots,a_n \in \prod_{i\in I} A_i$ then

$$t^{\mathbf{P}^*(\prod_{i\in I}\mathbf{A}_i)}(a_1,\ldots,a_n) = \prod_{i\in I} t^{\mathbf{P}^*(\mathbf{A}_i)}(a_1(i),\ldots,a_n(i)).$$

- The direct product of a family of multialgebras which satisfy a certain identity (weak or strong) satisfies the same identity.

Direct limits of direct systems of multialgebras

Let $\mathcal{A} = ((\mathbf{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ be a direct system of multialgebras and let $A_\infty = \{\widehat{x} \mid x \in A\}$ be the direct limit of the direct system of their underlying sets (A is the disjoint union of the sets A_i). Let $n \in \mathbb{N}$ and $f \in \mathcal{F}_n$. If $\widehat{x_1}, \ldots, \widehat{x_n} \in A_\infty$ and for any $j \in \{1, ..., n\}$ we consider that $x_j \in A_{i_j}$ ($i_j \in I$) then

$$f^{\mathbf{A}_\infty}(\widehat{a_1}, \ldots, \widehat{a_n}) = \{\widehat{a} \in A_\infty \mid \exists m \in I, i_1 \leq m, \ldots, i_n \leq m,$$
$$a \in f^{\mathbf{A}_m}(\varphi_{i_1 m}(a_1), \ldots, \varphi_{i_n m}(a_n))\}.$$

$\mathbf{A}_\infty = (A_\infty, F)$ is a multialgebra, called *the direct limit of* $\mathcal{A}$.

- If t is an n-ary term of type $\mathcal{F}$, $a_1, \ldots, a_n \in A$ and $i_1, \ldots, i_n \in I$ are such that $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ then

$$t^{\mathbf{P}^*(\mathbf{A}_\infty)}(\widehat{a_1}, \ldots, \widehat{a_n}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\ i_1 \leq m, \ldots,\ i_n \leq m,$$
$$a \in t^{\mathbf{P}^*(\mathbf{A}_m)}(\varphi_{i_1 m}(a_0), \ldots, \varphi_{i_n m}(a_n))\}.$$

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Factor multialgebras which are universal algebras

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$$f^{\mathbf{A}}(x_1, \ldots, x_n)\overline{\overline{\rho}}f^{\mathbf{A}}(y_1, \ldots, y_n);$$

d) *for any* n*-ary term* t *of type* $\mathcal{F}$, *and any* $x_i, y_i \in A$ *with* $x_i \rho y_i$ $(i \in \{1, \ldots, n\})$

$$t^{\mathbf{P}^*(\mathbf{A})}(x_1, \ldots, x_n)\,\overline{\overline{\rho}}\,t^{\mathbf{P}^*(\mathbf{A})}(y_1, \ldots, y_n).$$

We denote by $E_{ua}(\mathbf{A})$ *the set of the relations characterized above.*

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- If $\rho \in E_{ua}(\mathbf{A})$ then every identity which holds in $\mathbf{A}$ also holds in the algebra $\mathbf{A}/\rho$. In particular, any (weak or strong) identity which holds on $\mathbf{A}$ is also satisfied in $\overline{\mathbf{A}}$.

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- (Pelea) $\alpha^*_{\mathbf{A}}$ is the transitive closure of the relation α defined by

$$x \alpha_{\mathbf{A}} y \ \Leftrightarrow\ x, y \in t^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_n)$$

for some n-ary term t of type $\mathcal{F}$ and some $a_1, \dots, a_n \in A$.

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- (Pelea, Purdea) If $\mathbf{B}$ is a universal algebra of type $\mathcal{F}$, ρ is an equivalence relation on B, $\theta(\rho)$ is the congruence of $\mathbf{B}$ generated by ρ, t is an n-ary term of type $\mathcal{F}$, and $x, y, z_1, \ldots, z_n \in B$ then

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$\Rightarrow$ (Pelea, Purdea) $\overline{\mathbf{B}/\rho} \cong \mathbf{B}/\theta(\rho)$.

The fundamental relation of a hypergroup

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- The fundamental algebra of a hypergroup is a group.

Example

Let $(G, \cdot)$ be a group, $H \leq G$, $G/H = \{xH \mid x \in G\}$ and

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

The hypergroupoid $(G/H, \cdot)$ is a hypergroup (Marty).

If $\overline{G/H}$ is the fundamental group of the hypergroup G/H and $\overline{H}$ is the smallest normal subgroup of G which contains H then

$$\overline{G/H} \cong G/\overline{H}.$$

The fundamental algebra of the direct limit

- (Pelea) Let **A**, **B** be multialgebras, $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ their fundamental algebras and φ_A, φ_B the canonical projections. For any homomorphism $f : A \to B$ there exists only one universal algebra homomorphism $\overline{f} : \overline{A} \to \overline{B}$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ \overline{A} \arrow[r, "\overline{f}"] & \overline{B} \end{tikzcd}$$

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$\Rightarrow$ (Pelea) The factorization modulo the fundamental relation defines a functor G from the category of $Malg(\mathcal{F})$ of the multialgebras of type $\mathcal{F}$ into the category $Alg(\mathcal{F})$ of the universal algebras of the same type which is a left adjoint for the inclusion functor $U : Alg(\mathcal{F}) \longrightarrow Malg(\mathcal{F})$.

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$\Rightarrow$ (Pelea) The fundamental algebra of the direct limit of a direct system of multialgebras is (isomorphic to) the direct limit of their fundamental algebras.

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- Example

  Take the hypergroupoids $(H_1, \circ)$ and $(H_2, \circ)$ given by the following tables:

H_1	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

H_2	x	y	z
x	x	y, z	y, z
y	y, z	y, z	y, z
z	y, z	y, z	y, z

  $\overline{H_1 \times H_2}$ has 8 elements, while $\overline{H_1} \times \overline{H_2}$ has only 6 elements.

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- (Pelea) The functor $G : HG \to Grp$ preserves the finite products.
- (Pelea) The fundamental group of the direct product of the hypergroups $((H_i, \cdot) \mid i \in I)$ is (isomorphic to) the direct product of the fundamental groups $((\overline{H_i}, \cdot) \mid i \in I)$ if and only if there exists $n \in \mathbb{N}^*$ such that $\beta^{H_i} = \beta_n^{H_i}$ for all i's, with the exception of a finite number.