On some category theory constructions of multialgebras

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The objects

Let $\mathcal{F}$ be a type of (multi)algebras. *A* $\mathcal{F}$*-multialgebra* $\mathbf{A} = (A, F)$ consists of a set A and a family of multioperations F obtained by associating a multioperation $f^{\mathbf{A}}$ (or, simply, f) on A to each symbol f from $\mathcal{F}$.

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$\Rightarrow$ the factors of universal algebras modulo equivalence relations (which are not necessarily congruences)

(F. Marty, 1934) Let $(G, \cdot)$ be a group, $H \leq G$ and $G/H = \{xH \mid x \in G\}$. The equality

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

defines a binary multioperation

$$G/H \times G/H \to P^*(G/H)$$

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(Grätzer, 1962) Any multialgebra **A** results from a universal algebra **B** and an appropriate equivalence ρ of B as before, i.e. by taking

$$f(a_1/\rho, \ldots, a_n/\rho) = \{b/\rho \mid b = f(b_1, \ldots, b_n), a_i \rho b_i, i = 1, \ldots, n\}.$$

The morphisms

Let $\mathbf{A} = (A, F)$, $\mathbf{B} = (B, F)$ be $\mathcal{F}$-multialgebras and $h : A \to B$.

- h is a *multialgebra homomorphism* if for any $n \in \mathbb{N}$, any $f \in \mathcal{F}_n$ and any $a_1, \ldots, a_n \in A$,

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The *multialgebra isomorphisms* are the bijective ideal homomorphisms.

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Preservation properties of these functors with respect to products and directed limits and colimits

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$$x \alpha_{\mathbf{A}} y \Leftrightarrow \exists n \in \mathbb{N}, \exists t \in \mathrm{Clo}_n(\mathbf{P}^*(\mathbf{A})), \exists a_1, \ldots, a_n \in A :$$
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The category of $\mathcal{F}$-**Alg** is a reflective subcategory of the category $\mathcal{F}$-**Malg** and the fundamental functor is a reflector for $\mathcal{F}$-**Alg**.

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$\Rightarrow$ the above results can be rephrased for particular multialgebras such as: semihypergroups, hypergroups or hyperrings.

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Yet, the fundamental functor from the category of hypergroups into the category of groups preserves the finite products, and similar results hold for some particular categories of hyperrings.

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Let $(I, \leq)$ be a directed set,

$$D : (I, \geq) \to \mathbf{Malg}(\tau),\ D(i) = \mathbf{D}_i,\ D(i \to j) = \varphi_j^i$$

an inverse system of $\mathcal{F}$-multialgebras,

$$D_\infty = \{(a_i)_{i\in I} \in \prod_{i\in I} D_i \mid \forall j, k \in I,\ j \leq k,\ \varphi_j^k(a_k) = a_j\}$$

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If $f \in \mathcal{F}_n$, the corresponding "multioperations" should be defined on D_∞ by the equalities

$$f^{\mathbf{D}_\infty}((a_i^1)_{i\in I}, \ldots, (a_i^n)_{i\in I}) = \prod_{i\in I} f^{\mathbf{D}_i}(a_i^1, \ldots, a_i^n) \cap D_\infty = \varprojlim f^{\mathbf{D}_i}(a_i^1, \ldots, a_i^n).$$

An example of directed limit of monounary multialgebras which is a relational system which is not a multialgebra

Higman and Stone (1954) gave an example of an inverse system of (countable) sets, with surjective mappings and empty inverse limit: Let ω_1 be the first uncountable ordinal and for $\alpha < \omega_1$,

$$E_\alpha = \{\gamma \mid \gamma \leq \alpha\},\ F_\alpha = \{g \in \mathbb{R}^{E_\alpha} \mid g \text{ is strictly increasing}\};$$

and for $\alpha < \beta < \omega_1$, let

$$\theta_\alpha^\beta \colon F_\beta \to F_\alpha,\ \theta_\alpha^\beta(g) = g|_{E_\alpha} \text{ (the restriction of } g \text{ to } E_\alpha).$$

Higman and Stone define by transfinite induction a family of subsets S_α of F_α for which

$$|S_\alpha| = \aleph_0 \text{ and } \theta_\alpha^\beta(S_\beta) = S_\alpha \text{ whenever } \alpha < \beta,$$

such that the inverse system $(S_\alpha \mid \alpha < \omega_1)$ (with the corresponding restrictions of the functions θ_α^β) has the desired property.

An example ...

Starting from Higman and Stone example, we take $D_\alpha = S_\alpha \cup \{0_{E_\alpha}\}$ for each $\alpha < \omega_1$, where $0_{E_\alpha} : E_\alpha \to \mathbb{R},\ 0_{E_\alpha}(\gamma) = 0$, and

$$f^{D_\alpha} : D_\alpha \to P^*(D_\alpha),\ f^{D_\alpha}(g) = S_\alpha,\ \forall g \in D_\alpha.$$

Consider

$$\varphi_\alpha^\beta \colon D_\beta \to D_\alpha,\ \varphi_\alpha^\beta(g) = g|_{E_\alpha}\ (\alpha < \beta).$$

Clearly, $\varphi_\alpha^\beta|_{S_\alpha} = \theta_\alpha^\beta|_{S_\alpha}$ and φ_α^β are (ideal) homomorphisms, thus we obtain an inverse system of monounary multialgebras.

The set D_∞ is not empty since $(0_{E_\alpha})_{\alpha<\omega_1} \in D_\infty$ but

$$f^{\mathbf{D}_\infty}((0_{E_\alpha})_{\alpha<\omega_1}) = \varprojlim f^{\mathbf{D}_\alpha}(0_{E_\alpha}) = \varprojlim S_\alpha = \emptyset.$$

For details, see ...

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The participation to the conference was supported by the CNCS-UEFISCDI grant PN-II-ID-PCE-2012-4-0100.