On some particular equivalence relations of multialgebras

Cosmin Pelea

September 25-28, 2013

The genesis:

(F. Marty, 8th Congress of the Scandinavian Mathematicians, Stockholm, 1934)

Let $(G, \cdot)$ be a group, $H \leq G$ and $G/H = \{xH \mid x \in G\}$. The equality

$$(1) \qquad (xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

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defines an operation on G/H if and only if $H \trianglelefteq G$.

In general, (1) defines a function

$$G/H \times G/H \to P^*(G/H)$$

called binary multioperation (on G/H) (and $(G/H, \cdot)$ is a hypergroup).

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- Let $\mathcal{F}$ be a type of (multi)algebras. A multialgebra $\mathbf{A} = (A, F)$ of type $\mathcal{F}$ consists of a set A and a family of multioperations F obtained by associating a multioperation $f^{\mathbf{A}}$ (or, simply, f) on A to each symbol f from $\mathcal{F}$.

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- (Grätzer, 1962) *Any multialgebra* $\mathbf{A}$ *results from a universal algebra* $\mathbf{B}$ *and an appropriate equivalence* ρ *of* B *as before, i.e. by taking*

$$f(a_1/\rho, \ldots, a_n/\rho) = \{b/\rho \mid b = f(b_1, \ldots, b_n), a_i \rho b_i, i = 1, \ldots, n\}.$$

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- For any multialgebra $\mathbf{A}$ of type $\mathcal{F}$ and any equivalence relation ρ on A, the factor multialgebra $\mathbf{A}/\rho$ (determined on $\mathbf{A}$ by ρ) is defined by
$$f(a_1/\rho, \ldots, a_n/\rho) = \{b/\rho \mid b \in f(b_1, \ldots, b_n), a_i \rho b_i, i = 1, \ldots, n\}.$$

A possible start:

M. Dresher, O. Ore, 'Theory of multigroups', *Amer. J. Math.*, **60** 1938, 705–733.

M. Koskas, 'Groupoïdes, demi-hypergroupes et hypergroupes', *J. Math. Pures et Appl.*, **49**, 1970, 155–192.

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- the equivalence relations of (semi)hypergroups for which the factor hypergroup(oid) is a group (strongly regular equivalences)
- the smallest strongly regular equivalence of a (semi)hypergroup for which the factor (hyper)group is commutative

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2. Determine the smallest equivalence relation of a (general) multialgebra $\mathbf{A}$ for which the factor multialgebra is in a certain variety of universal algebras.
3. Find universal algebra isomorphisms in order to obtain easier ways to construct the corresponding factor multialgebras.

The path:

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- a characterization of the smallest equivalence relation of $\mathbf{A}$ containing a given relation which provides a factor multialgebra which is universal algebra (2013)
- some isomorphism theorems (2006, 2013)

The tools:

- The universal algebra $\mathbf{P}^*(\mathbf{A})$ of the nonempty subsets of $\mathbf{A}$, given by

$$f^{\mathbf{P}^*(\mathbf{A})}(A_1, \ldots, A_n) = \bigcup\{f^{\mathbf{A}}(a_1, \ldots, a_n) \mid \ a_i \in A_i,\ i = 1, \ldots, n\}.$$

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For any $p\in\mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$, there exist $m\in\mathbb{N}$, $m\geq 1$, $b_1,\dots,b_m\in A$ and $t\in\mathrm{Clo}(\mathbf{P}^*(\mathbf{A}))$ such that

$$p^{\mathbf{P}^*(\mathbf{A})}(A_1)=t^{\mathbf{P}^*(\mathbf{A})}(A_1,b_2,\dots,b_m),\ \forall A_1\in P^*(A).$$

The results:

Proposition (Pelea, Purdea, 2006)

For an equivalence ρ of a multialgebra $\mathbf{A}$, the following conditions are equivalent:
(a) $\mathbf{A}/\rho$ is a universal algebra ($\rho \in E_{ua}(\mathbf{A})$);

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(b) *if* $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $a, b, x_1, \ldots, x_n \in A$, $a\rho b$, *then, for all* $i = 1, \ldots, n$,

$$f^{\mathbf{A}}(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)\overline{\overline{\rho}}f^{\mathbf{A}}(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n);$$

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(c) *if* $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $x_i, y_i \in A$ *and* $x_i \rho y_i$ $(i = 1, \ldots, n)$, *then*

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(d) *if $m \in \mathbb{N}$, t is an m-ary term, $x_i, y_i \in A$, $x_i\rho y_i$ $(i = 1, \ldots, m)$, then*

$$t^{\mathbf{P}^*(\mathbf{A})}(x_1, \ldots, x_m)\,\overline{\overline{\rho}}\,t^{\mathbf{P}^*(\mathbf{A})}(y_1, \ldots, y_m);$$

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(*e*) *if* $p \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$, $x, y \in A$ *and* $x\rho y$ *then* $p(x)\overline{\overline{\rho}}\,p(y)$.

The results:

The poset $\langle E_{ua}(\mathbf{A}), \subseteq\rangle$ is an algebraic closure system on $A \times A$. Let $\alpha^{\mathbf{A}}$ be the corresponding closure operator.

Theorem (Pelea, Purdea, Stanca)

If $\mathcal{I} = \{q_i = r_i \mid i \in I\}$ ($I \neq \emptyset$, q_i, r_i m_i-ary terms of type $\mathcal{F}$, $i \in I$) and $\mathbf{A} = \langle A, F\rangle$ is an $\mathcal{F}$-multialgebra, the smallest equivalence relation on $\mathbf{A}$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I}$ is the transitive closure $\alpha^*_{\mathcal{I}}$ of the relation $\alpha_{\mathcal{I}} \subseteq A \times A$ defined by:

$$x \alpha_{\mathcal{I}} y \ \Leftrightarrow\ \exists i \in I,\ \exists n_i \in \mathbb{N}^*,\ \exists p_i \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A})),\ \exists a^i_1, \ldots, a^i_{m_i} \in A:$$
$$x \in p_i^{\mathbf{P}^*(\mathbf{A})}(q_i^{\mathbf{P}^*(\mathbf{A})}(a^i_1, \ldots, a^i_{m_i})),\ y \in p_i^{\mathbf{P}^*(\mathbf{A})}(r_i^{\mathbf{P}^*(\mathbf{A})}(a^i_1, \ldots, a^i_{m_i})),$$
$$\text{or}\ \ y \in p_i^{\mathbf{P}^*(\mathbf{A})}(q_i^{\mathbf{P}^*(\mathbf{A})}(a^i_1, \ldots, a^i_{m_i})),\ x \in p_i^{\mathbf{P}^*(\mathbf{A})}(r_i^{\mathbf{P}^*(\mathbf{A})}(a^i_1, \ldots, a^i_{m_i})).$$

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Theorem (Pelea, 2013)

Let $\mathbf{A} = \langle A, F\rangle$ be a multialgebra of type $\mathcal{F}$ and $\theta \subseteq A \times A$. The relation $\alpha^{\mathbf{A}}(\theta)$ is defined as follows: $\langle x, y\rangle \in \alpha^{\mathbf{A}}(\theta)$ if and only if there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$ of elements from A, some pairs $\langle b_1, c_1\rangle, \ldots, \langle b_k, c_k\rangle \in \theta$ and some unary polynomial functions $p_1, \ldots, p_k$ from $\mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ such that for all $i \in \{1, \ldots, k\}$,

$$\langle t_{i-1}, t_i\rangle \in p_i(b_i) \times p_i(c_i) \text{ or } \langle t_i, t_{i-1}\rangle \in p_i(b_i) \times p_i(c_i).$$

The results:

The previous result follows from the next one by taking

$$\theta = \bigcup\{q_i^{\mathbf{P}^*(\mathbf{A})}(a_1,\dots,a_{m_i}) \times r_i^{\mathbf{P}^*(\mathbf{A})}(a_1,\dots,a_{m_i}) \mid a_1,\dots,a_{m_i} \in A, i \in I\}.$$

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The results:

Let $\mathcal{I}$ be a set of identities.

Theorem (Pelea, Purdea, Stanca)

The variety $M(\mathcal{I})$ of the $\mathcal{F}$-algebras which satisfy all the identities from $\mathcal{I}$ is a reflective subcategory of the category of $\mathcal{F}$-multialgebras.

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Corollary

The factorization of $\mathcal{F}$-multialgebras modulo $\alpha^*_{\mathcal{I}}$ provides a functor which is a reflector for $M(\mathcal{I})$.

The results:

Let I, J be two disjoint nonempty sets, q_i, r_i m_i-ary terms $(i \in I \cup J)$, $\mathcal{I} = \{q_i = r_i \mid i \in I\}$ and $\mathcal{J} = \{q_i = r_i \mid i \in J\}$.

Theorem (Pelea, 2013)

If $\mathbf{A} = \langle A, F\rangle$ is a multialgebra of type $\mathcal{F}$ then

$$\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong (\mathbf{A}/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}}$$

($\underline{\alpha}^*_{\mathcal{I}}$ is the smallest congruence of $\mathbf{A}/\alpha^*_{\mathcal{J}}$ for which the factor algebra satisfies the identities from $\mathcal{I}$).

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For $\mathcal{J} = \{\mathrm{x} = \mathrm{x}\}$, $\alpha^*_{\mathcal{J}} = \alpha^*_{\mathbf{A}}$ is the fundamental relation of $\mathbf{A}$, thus

Corollary

If $\mathbf{A} = \langle A, F\rangle$ is a $\mathcal{F}$-multialgebra, then $\mathbf{A}/\alpha^*_{\mathcal{I}} \cong (\mathbf{A}/\alpha^*_{\mathbf{A}})/\underline{\alpha}^*_{\mathcal{I}}$.

Examples: Hypergroups:

A multialgebra $\langle H, \cdot \rangle$ with one binary associative multioperation is called semihypergroup. A hypergroup is a nonempty semihypergroup $\langle H, \cdot \rangle$ which satisfies the condition

$$a \cdot H = H \cdot a = H, \ \forall a \in H. \tag{2}$$

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In a hypergroup $\langle H, \cdot \rangle$, the condition (2) defines two binary multioperations $/, \backslash : H \times H \to P^*(H)$ as follows

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$$b/a = \{x \in H \mid b \in x \cdot a\}, \ a \backslash b = \{x \in H \mid b \in a \cdot x\},$$

but

$$E_{ua}(\langle H, \cdot \rangle) = E_{ua}(\langle H, \cdot, /, \backslash \rangle)$$

and $\langle H/\rho, \cdot \rangle$ is a group for any $\rho \in E_{ua}(\langle H, \cdot \rangle)$.

Hypergroups:

Let $\langle H, \cdot\rangle$ be a hypergroup.

(Freni, 2002) the smallest equivalence of H for which the factor of $\langle H, \cdot\rangle$ is a commutative group is (the transitive closure γ^* of)

$$x\gamma y \Leftrightarrow \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

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where $\gamma_1 = \delta_H$ and for $n > 1$,

$$\begin{aligned} x\gamma_n y \Leftrightarrow \exists z_1, \ldots, z_n \in H,\ \exists i \in \{1, \ldots, n-1\} :& \\ x \in z_1 \cdots z_{i-1}(z_i z_{i+1}) z_{i+2} \cdots z_n,& \\ y \in z_1 \cdots z_{i-1}(z_{i+1} z_i) z_{i+2} \cdots z_n.& \end{aligned}$$

The transitive closure of $\alpha_{\{\mathrm{x}_1\mathrm{x}_2=\mathrm{x}_2\mathrm{x}_1\}}$ coincides with the smallest equivalence of H for which the factor of $\langle H, \cdot \rangle$ is a commutative group is (the transitive closure γ^* of)

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since the cycles $(1,2), (2,3), \ldots, (n-1, n)$ generate the symmetric group $(S_n, \circ)$.

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Let $\langle H, \cdot \rangle$ be a (semi)hypergroup.

(Koskas, 1970) The smallest strongly regular equivalence of $\langle H, \cdot \rangle$ (the fundamental relation of $\langle H, \cdot \rangle$) is the transitive closure β^* of

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For any hypergroup $\langle H, \cdot \rangle$,

$$H/\gamma \cong (H/\beta)/(H/\beta)'.$$

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The multialgebra $\langle A, +, \cdot\rangle$ is a hyperring if $\langle A, +\rangle$ is a hypergroup, $\langle A, \cdot\rangle$ is a semihypergroup and $\cdot$ is distributive with respect to $+$.

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$\alpha^*_{\mathcal{I}}$ is the transitive closure α^* of the relation consisting of all the pairs $\langle x, y\rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, a permutation $\tau \in S_n$, $x_{i1}, \ldots, x_{ik_i} \in A$, and $\sigma_i \in S_{k_i}(i = 1, \ldots, n)$ such that

$$x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right) \text{ and } y \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{\tau(i)\sigma_{\tau(i)}(j)}\right)$$

(Davvaz, Vougiouklis, 2007).

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then $\alpha^*_{\mathcal{I}}$ is the smallest equivalence relation on A for which the factor multialgebra is a commutative ring.
$\alpha^*_{\mathcal{I}}$ is the transitive closure of the union of all the Cartesian products

$$t(a_1,\dots,a_m)\times t'(a_1,\dots,a_m),$$

where $t(a_1,\dots,a_m)$ is a sum of products of elements of A (we allow the sum to have only one term and the products to have only one factor), and $t'(a_1,\dots,a_m)$ is obtained from $t(a_1,\dots,a_m)$ either by permuting two consecutive factors in a product or by permuting two consecutive terms in the sum.

Hyperrings:

The fundamental algebra $\langle \overline{A}, +, \cdot \rangle$ of a hyperring $\langle A, +, \cdot \rangle$ is a distributive nearring, hence

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For a hyperring $\langle A, +, \cdot \rangle$ with $+$ weak commutative or with a multiplicative identity, the fundamental relation of $\langle A, +, \cdot \rangle$ and $\alpha^*_{\{\mathrm{x}_1+\mathrm{x}_2=\mathrm{x}_2+\mathrm{x}_1\}}$ coincide,

$$\alpha^* = \alpha^*_{\{\mathrm{x}_1\mathrm{x}_2=\mathrm{x}_2\mathrm{x}_1\}}$$

and the ring $\langle A/\alpha^*, +, \cdot \rangle$ is isomorphic to the factor of the fundamental ring of $\langle A, +, \cdot \rangle$ over its commutator ideal.

Hyperrings:

For a hyperring $\langle A, +, \cdot \rangle$ with $\cdot$ operation,

$$\alpha^*_{\{\mathrm{x}_1+\mathrm{x}_2=\mathrm{x}_2+\mathrm{x}_1\}} = \gamma$$

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For a distributive nearring $\langle R, +, \cdot \rangle$, the derived subgroup R' of $\langle R, + \rangle$ is an ideal.

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For a distributive nearring $\langle R, +, \cdot\rangle$, the derived subgroup R' of $\langle R, +\rangle$ is an ideal.

For any hyperring $\langle A, +, \cdot\rangle$,

$$A/\alpha^*_{\{\mathrm{x}_1+\mathrm{x}_2=\mathrm{x}_2+\mathrm{x}_1\}} \cong \overline{A}/\overline{A}'.$$

For details, see ...

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