Fundamental relations in multialgebras. A survey

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commutative $\sim$ *of a (semi)hypergroup* $=$ the smallest equivalence relation of a (semi)hypergroup for which the factor semihypergroup is a commutative (semi)group;

$\mathcal{I}-\sim$ *of a multialgebra* $=$ the smallest equivalence relation of a multialgebra for which the factor multialgebra is a universal algebra satisfying a given set of identities $\mathcal{I}$.

Some tools:

- The universal algebra $\mathbf{P}^*(\mathbf{A})$ of the nonempty subsets of $\mathbf{A}$, given by

$$f^{\mathbf{P}^*(\mathbf{A})}(A_1, \ldots, A_n) = \bigcup \{ f^{\mathbf{A}}(a_1, \ldots, a_n) \mid \ a_i \in A_i, \ i = 1, \ldots, n \}.$$

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- The subuniverse $\mathrm{Pol}_m^A(\mathbf{P}^*(\mathbf{A}))$ of $\mathrm{Pol}_m(\mathbf{P}^*(\mathbf{A}))$ generated by $\{c_a^m\mid a\in A\}\cup\{e_i^m\}$, where $c_a^m,e_i^m:P^*(A)^m\to P^*(A)$ are given by

$$c_a^m(A_1,\ldots,A_m)=\{a\}\text{ and }e_i^m(A_1,\ldots,A_m)=A_i.$$

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- The clone $\mathrm{Clo}(\mathbf{P}^*(\mathbf{A}))$ of the term functions of $\mathbf{P}^*(\mathbf{A})$.

Fundamental relation (specific)

(P., 2001) The fundamental relation $\alpha^*_{\mathbf{A}}$ of the multialgebra $\mathbf{A}$ is the transitive closure of the relation $\alpha_{\mathbf{A}}$ defined as follows

$$x \alpha_{\mathbf{A}} y \Leftrightarrow \exists n \in \mathbb{N}, \exists p \in \mathrm{Pol}_n^A(\mathbf{P}^*(\mathbf{A})), \exists a_1, \ldots, a_n \in A : \\ x, y \in p(a_1, \ldots, a_n).$$

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For any $p \in \mathrm{Pol}^A_1(\mathbf{P}^*(\mathbf{A}))$, there exist $m \in \mathbb{N}$, $m \geq 1$, $b_1, \ldots, b_m \in A$ and $t \in \mathrm{Clo}(\mathbf{P}^*(\mathbf{A}))$ such that

$$p^{\mathbf{P}^*(\mathbf{A})}(A_1) = t^{\mathbf{P}^*(\mathbf{A})}(A_1, b_2, \ldots, b_m), \ \forall A_1 \in P^*(A).$$

Fundamental relation (generic)

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(b) if $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $a, b, x_1, \dots, x_n \in A$, $a\rho b$, then, for all $i = 1, \dots, n$,

$$f^{\mathbf{A}}(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n)\overline{\overline{\rho}}f^{\mathbf{A}}(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n);$$

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$$p(x_1, \ldots, x_m) \, \overline{\overline{\rho}} \, p(y_1, \ldots, y_m);$$

(e) if $m \in \mathbb{N}$, t is an m-ary term, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then

$$t^{\mathbf{P}^*(\mathbf{A})}(x_1, \ldots, x_m) \, \overline{\overline{\rho}} \, t^{\mathbf{P}^*(\mathbf{A})}(y_1, \ldots, y_m);$$

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(f) if $p \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$, $x, y \in A$ and $x\rho y$ then

$$p(x)\overline{\overline{\rho}}\,p(y).$$

The $\mathcal{I}$-fundamental relation

(P., Purdea, 2006) α^*_{qr} = the smallest relation from $E_{ua}(\mathbf{A})$ for which the factor (multi)algebra satisfies the identity $q = r$

The $\mathcal{I}$-fundamental relation

Theorem (P., Purdea, Stanca)

*Let $\mathcal{I} = \{q_i = r_i \mid i \in I\}$ ($I \neq \emptyset$, q_i, r_i m_i-ary terms of type $\mathcal{F}$, $i \in I$) and let $\mathbf{A} = \langle A, F\rangle$ be an $\mathcal{F}$-multialgebra. The smallest equivalence relation on $\mathbf{A}$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I}$ (i.e. the $\mathcal{I}$-*fundamental relation of $\mathbf{A}$*) is the transitive closure $\alpha_{\mathcal{I}}^*$ of the relation $\alpha_{\mathcal{I}} \subseteq A \times A$ defined by:*

$$\begin{aligned} x\alpha_{\mathcal{I}}y \;\Leftrightarrow\; & \exists i \in I,\ \exists p_i \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A})),\ \exists a_1^i, \ldots, a_{m_i}^i \in A: \\ & x \in p_i^{\mathbf{P}^*(\mathbf{A})}(q_i^{\mathbf{P}^*(\mathbf{A})}(a_1^i, \ldots, a_{m_i}^i)),\ y \in p_i^{\mathbf{P}^*(\mathbf{A})}(r_i^{\mathbf{P}^*(\mathbf{A})}(a_1^i, \ldots, a_{m_i}^i)), \\ \text{or}\ \ & y \in p_i^{\mathbf{P}^*(\mathbf{A})}(q_i^{\mathbf{P}^*(\mathbf{A})}(a_1^i, \ldots, a_{m_i}^i)),\ x \in p_i^{\mathbf{P}^*(\mathbf{A})}(r_i^{\mathbf{P}^*(\mathbf{A})}(a_1^i, \ldots, a_{m_i}^i)). \end{aligned}$$

(P., Purdea, 2006) $\alpha_{qr}^* =$ the smallest relation from $E_{ua}(\mathbf{A})$ for which the factor (multi)algebra satisfies the identity $q = r$

$$\Longrightarrow\ \alpha_{\mathcal{I}}^* = \bigvee_{i\in I} \alpha_{q_i r_i}^*.$$

The general recipe

For an $\mathcal{F}$-multialgebra $\mathbf{A} = \langle A, F \rangle$, the poset $\langle E_{ua}(\mathbf{A}), \subseteq \rangle$ is an algebraic closure system on $A \times A$.

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For an $\mathcal{F}$-multialgebra $\mathbf{A} = \langle A, F\rangle$, the poset $\langle E_{ua}(\mathbf{A}), \subseteq\rangle$ is an algebraic closure system on $A \times A$. If $\alpha^{\mathbf{A}}$ is the corresponding closure operator then

Theorem (Pelea, 2013)

If $\theta \subseteq A \times A$, the relation $\alpha^{\mathbf{A}}(\theta)$ is defined as follows: $\langle x, y\rangle \in \alpha^{\mathbf{A}}(\theta)$ if and only if there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$ of elements from A, some pairs $\langle b_1, c_1\rangle, \ldots, \langle b_k, c_k\rangle \in \theta$ and some unary polynomial functions $p_1, \ldots, p_k$ from $\mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ such that for all $i \in \{1, \ldots, k\}$,

$$\langle t_{i-1}, t_i\rangle \in p_i(b_i) \times p_i(c_i) \text{ or } \langle t_i, t_{i-1}\rangle \in p_i(b_i) \times p_i(c_i).$$

The general recipe

For an $\mathcal{F}$-multialgebra $\mathbf{A} = \langle A, F\rangle$, the poset $\langle E_{ua}(\mathbf{A}), \subseteq\rangle$ is an algebraic closure system on $A \times A$. If $\alpha^{\mathbf{A}}$ is the corresponding closure operator then the $\mathcal{I}$-fundamental relation is $\alpha^{\mathbf{A}}(R_{\mathcal{I}})$, where

$$R_{\mathcal{I}} = \bigcup\{q_i^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_{m_i}) \times r_i^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_{m_i}) \mid a_1, \dots, a_{m_i} \in A, i \in I\}$$

thus $\alpha^*_{\mathcal{I}}$ follows from the next result by taking $\theta = R_{\mathcal{I}}$.

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Applications...

...to hypergroups

Let $\langle H, \cdot \rangle$ be a hypergroup.

$E_{ua}(\langle H, \cdot \rangle)$ = the set of the strongly regular equivalence relations

The reproducibility condition defines the binary multioperations

$/, \backslash : H^2 \to P^*(H)$, $b/a = \{x \in H \mid b \in x \cdot a\}$, $a \backslash b = \{x \in H \mid b \in a \cdot x\}$,

but

$$E_{ua}(\langle H, \cdot \rangle) = E_{ua}(\langle H, \cdot, /, \backslash \rangle)$$

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(Koskas, 1970) The fundamental relation of $\langle H, \cdot \rangle$ is (the transitive closure β^* of) the relation β defined by

$$x \beta y \Leftrightarrow \exists n \in \mathbb{N}^*,\ \exists a_1, \ldots, a_n \in H :\ x, y \in a_1 \cdots a_n.$$

Applications. . .

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Let $\langle H, \cdot \rangle$ be a hypergroup.

(Freni, 2002) the commutative-fundamental relation of $\langle H, \cdot \rangle$ is (the transitive closure γ^* of) the relation γ defined by

$$x \gamma y \Leftrightarrow \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

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$$\alpha_{\{\mathrm{x}_1\mathrm{x}_2=\mathrm{x}_2\mathrm{x}_1\}} = \bigcup_{n\in\mathbb{N}^*} \gamma_n,$$

where $\gamma_1 = \delta_H$ and for $n > 1$,

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since the cycles (1 2),(2 3),. . . ,($n-1$ n) generate the symmetric group S_n

Applications. . .

. . . to n-(semi)hypergroups

Let $\langle H, f\rangle$ be an n-semihypergroup ($n \in \mathbb{N},\ n \geq 2$).

(Davvaz, Fotea, 2008) the form of the fundamental relation of $\langle H, f\rangle$

(Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f\rangle$

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Let $\langle H, f\rangle$ be an n-semihypergroup ($n \in \mathbb{N},\ n \geq 2$).

Using the associativity of f one obtains the form of the term functions from $\langle P^*(H), f\rangle$ and from the characterization of the fundamental relation of a (general) multialgebra one can immediately deduce (Davvaz, Fotea, 2008) the form of the fundamental relation of $\langle H, f\rangle$

If we take $\mathcal{I} = \{f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(n)}) \mid \sigma \in S_n\}$, then $\alpha^*_{\mathcal{I}}$ is (Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f\rangle$

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If we take $\mathcal{I} = \{f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(n)}) \mid \sigma \in S_n\}$, then $\alpha^*_{\mathcal{I}}$ is (Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f\rangle$

- If $\langle H, f\rangle$ is n-hypergroup, it hides in its structure the multioperations

$$(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \mapsto \{x \mid b \in f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)\}$$

but in the corresponding fundamental algebra the associativity of f and the existence of a solution for (each of) the equation $b = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ implies its uniqueness, thus

$$E_{ua}(\langle R, f\rangle) = E_{ua}(\langle R, f, f_1, \ldots, f_n\rangle).$$

Applications...

...to hyperrings

Let $\langle A, +, \cdot \rangle$ be a hyperring.

(Vougiouklis, 1991) the fundamental relation Γ^* of $\langle A, +, \cdot \rangle$ is the transitive closure of the relation consisting of all the pairs $\langle x, y \rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, $x_{i1}, \ldots, x_{ik_i} \in A$ $(i = 1, \ldots, n)$ such that

$$x, y \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right).$$

Applications...

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Let $\langle A, +, \cdot \rangle$ be a hyperring.

Since $\cdot$ is (sub)distributive with respect to $+$, any image of a term function of $\langle P^*(A), +, \cdot \rangle$ is contained in a sum of products of nonempty subsets of A, thus

the fundamental relation Γ^* of $\langle A, +, \cdot \rangle$ is the transitive closure of the relation consisting of all the pairs $\langle x, y \rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, $x_{i1}, \ldots, x_{ik_i} \in A$ $(i = 1, \ldots, n)$ such that

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Applications. . .

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Let $\langle A, +, \cdot\rangle$ be a hyperring and $\mathcal{I} = \{\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1,\ \mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_1\}$.

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(P., Purdea, Stanca) $\alpha^*_{\mathcal{I}}$ is the transitive closure of the union of all the Cartesian products

$$t(a_1, \ldots, a_m) \times t'(a_1, \ldots, a_m),$$

where $t(a_1, \ldots, a_m)$ is a sum of products of elements of A (we allow the sum to have only one term and the products to have only one factor), and $t'(a_1, \ldots, a_m)$ is obtained from $t(a_1, \ldots, a_m)$ either by permuting two consecutive factors in a product or by permuting two consecutive terms in the sum.

Moreover. . .

Theorem (P., Purdea, Stanca)

The variety $M(\mathcal{I})$ of the $\mathcal{F}$-algebras which satisfy all the identities from $\mathcal{I}$ is a reflective subcategory of the category of $\mathcal{F}$-multialgebras.

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• The factorization of $\mathcal{F}$-multialgebras modulo $\alpha_{\mathcal{I}}^*$ provides a functor $F_{\mathcal{I}}$ which is a reflector for $M(\mathcal{I})$:

- $F_{\mathcal{I}}$ is defined by $F_{\mathcal{I}}(\mathbf{A}) = \mathbf{A}/\alpha_{\mathcal{I}}^*$ on objects,
- $F_{\mathcal{I}}$ assigns to each multialgebra homomorphism $h : A \to B$ the unique universal algebra homomorphism $\overline{h}$ which makes the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ A/\alpha_{\mathcal{I}}^* \arrow[r, "\overline{h}"] & B/\alpha_{\mathcal{I}}^* \end{tikzcd}$$

commutative (the vertical arrows are the canonical projections).

Other corollaries . . .

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- $\mathbf{A}/\alpha^*_{\mathcal{I}} \cong (\mathbf{A}/\alpha^*_{\mathbf{A}})/\underline{\alpha}^*_{\mathcal{I}}$.

- The commutative-fundamental group of a hypergroup is the abelianization of its fundamental group.

- For a hyperring $\langle A, +, \cdot \rangle$ with $+$ weak commutative or with a multiplicative identity, its fundamental relation is $\alpha^*_{\{\mathrm{x}_1+\mathrm{x}_2=\mathrm{x}_2+\mathrm{x}_1\}}$,

$$\alpha^* = \alpha^*_{\{\mathrm{x}_1\mathrm{x}_2=\mathrm{x}_2\mathrm{x}_1\}}$$

and the ring A/α^* is isomorphic to the factor of the fundamental ring of A over its commutator ideal.

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• Replacing F by $F_{\mathcal{I}}$ in the above statement opens up the possibility to define the $\mathcal{I}$-*fundamental algebra* of a fuzzy multialgebra for any set $\mathcal{I}$ of identities.

. . . ... and other applications

(P., Purdea, Stanca) If **A** is a multialgebra, $\mu : A \to [0,1]$ is a fuzzy submultialgebra of **A** and $\rho \in E_{ua}(\mathbf{A})$, then μ_ρ defined by

$$\mu_\rho(y/\rho) = \bigvee \{\mu(x) \mid \langle x, y \rangle \in \rho\}$$

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$\Longrightarrow$ $\mu_{\alpha^*_{\mathcal{I}}}$ is a fuzzy subalgebra of $\mathbf{A}/\alpha^*_{\mathcal{I}}$.

$\Longrightarrow$ $\mu_{\alpha^*_{\mathbf{A}}}$ is a fuzzy subalgebra of the fundamental algebra of **A**.

For details, see ...

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