Fundamental relations in multialgebras. A survey

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*)commutative* ~ of a (semi)hypergroup = the smallest equivalence relation of a (semi)hypergroup for which the factor semihypergroup is a commutative (semi)group;
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\textit{commutative} \sim \textit{of a (semi)hypergroup} = \text{the smallest equivalence relation of a (semi)hypergroup for which the factor semihypergroup is a commutative (semi)group};

\mathcal{I}\sim \textit{of a multialgebra} = \text{the smallest equivalence relation of a multialgebra for which the factor multialgebra is a universal algebra satisfying a given set of identities } \mathcal{I}.
Some tools:

- The universal algebra $P^*(A)$ of the nonempty subsets of $A$, given by
  $$f_{P^*(A)}(A_1, \ldots, A_n) = \bigcup \{ f^A(a_1, \ldots, a_n) \mid a_i \in A_i, \ i = 1, \ldots, n \}.$$
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- The algebra $\langle \text{Pol}_m(P^*(A)) , F \rangle$ of the $m$-ary polynomial functions of the algebra $P^*(A)$.
Some tools:

- The universal algebra $\mathbf{P}^*(A)$ of the nonempty subsets of $A$, given by
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  \]

- The algebra $\langle \text{Pol}_m(\mathbf{P}^*(A)), F \rangle$ of the $m$-ary polynomial functions of the algebra $\mathbf{P}^*(A)$.

- The subuniverse $\text{Pol}^A_m(\mathbf{P}^*(A))$ of $\text{Pol}_m(\mathbf{P}^*(A))$ generated by $\{c^m_a \mid a \in A\} \cup \{e^m_i\}$, where $c^m_a, e^m_i : \mathbf{P}^*(A)^m \to \mathbf{P}^*(A)$ are given by
  \[
  c^m_a(A_1, \ldots, A_m) = \{a\} \quad \text{and} \quad e^m_i(A_1, \ldots, A_m) = A_i.
  \]
Some tools:

- The universal algebra $P^*(A)$ of the nonempty subsets of $A$, given by
  \[ f^{P^*(A)}(A_1, \ldots, A_n) = \bigcup \{ f^A(a_1, \ldots, a_n) \mid a_i \in A_i, \ i = 1, \ldots, n \}. \]

- The algebra $\langle \text{Pol}_m(P^*(A)), F \rangle$ of the $m$-ary polynomial functions of the algebra $P^*(A)$.

- The subuniverse $\text{Pol}_m^A(P^*(A))$ of $\text{Pol}_m(P^*(A))$ generated by $\{ c^m_a \mid a \in A \} \cup \{ e^m_i \}$, where $c^m_a, e^m_i : P^*(A)^m \to P^*(A)$ are given by
  \[ c^m_a(A_1, \ldots, A_m) = \{ a \} \text{ and } e^m_i(A_1, \ldots, A_m) = A_i. \]

- The clone $\text{Clo}(P^*(A))$ of the term functions of $P^*(A)$. 

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Fundamental relations in multialgebras. A survey
Fundamental relation (specific)

(P., 2001) The fundamental relation $\alpha^*_A$ of the multialgebra $A$ is the transitive closure of the relation $\alpha_A$ defined as follows

$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists p \in \text{Pol}^A_n(P^*(A)), \exists a_1, \ldots, a_n \in A : x, y \in p(a_1, \ldots, a_n).$$
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$$x, y \in p(a_1, \ldots, a_n).$$

[ or ]

$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists t \in \text{Clo}_n(P^*(A)), \exists a_1, \ldots, a_n \in A :$$

$$x, y \in t^{P^*(A)}(a_1, \ldots, a_n).$$
**Fundamental relation (specific)**

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$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists p \in \text{Pol}_n^A(P^*(A)), \exists a_1, \ldots, a_n \in A : x, y \in p(a_1, \ldots, a_n).$$

[ or

$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists t \in \text{Cl}_n(P^*(A)), \exists a_1, \ldots, a_n \in A : x, y \in t(P^*(A))(a_1, \ldots, a_n).$$ ]

(P., Purdea, 2006) The relation $\alpha_A^*$ of the multialgebra $A$ is the transitive closure of the relation $\alpha_A$ defined as follows

$$x \alpha_A y \iff \exists p \in \text{Pol}_1^A(P^*(A)), \exists a \in A : x, y \in p(a).$$
**Fundamental relation (specific)**

(P., 2001) The fundamental relation $\alpha_*^A$ of the multialgebra $A$ is the transitive closure of the relation $\alpha_A$ defined as follows

$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists p \in \text{Pol}_n^A(P^*(A)), \exists a_1, \ldots, a_n \in A : x, y \in p(a_1, \ldots, a_n).$$

[ or

$$x \alpha_A y \iff \exists n \in \mathbb{N}, \exists t \in \text{Clo}_n(P^*(A)), \exists a_1, \ldots, a_n \in A : x, y \in t^{P^*(A)}(a_1, \ldots, a_n).$$]

(P., Purdea, 2006) The relation $\alpha_*^A$ of the multialgebra $A$ is the transitive closure of the relation $\alpha_A$ defined as follows

$$x \alpha_A y \iff \exists p \in \text{Pol}_1^A(P^*(A)), \exists a \in A : x, y \in p(a).$$

For any $p \in \text{Pol}_1^A(P^*(A))$, there exist $m \in \mathbb{N}, m \geq 1, b_1, \ldots, b_m \in A$ and $t \in \text{Clo}(P^*(A))$ such that

$$p^{P^*(A)}(A_1) = t^{P^*(A)}(A_1, b_2, \ldots, b_m), \forall A_1 \in P^*(A).$$
Fundamental relation (generic)

(P., Purdea, 2006) For an equivalence relation $\rho$ of a multialgebra $A$, the following conditions are equivalent:

(a) $A/\rho$ is a universal algebra ($\rho \in E_{ua}(A)$);

(b) if $n \in \mathbb{N}$, $f \in F_n$, $a, b, x_1, \ldots, x_n \in A$, $a \rho b$, then, for all $i = 1, \ldots, n$,

$$f_A(x_1, \ldots, x_i-1, a, x_i+1, \ldots, x_n) \rho f_A(x_1, \ldots, x_i-1, b, x_i+1, \ldots, x_n)$$

(c) if $n \in \mathbb{N}$, $f \in F_n$, $x_i, y_i \in A$ and $x_i \rho y_i$ ($i = 1, \ldots, n$), then

$$f_A(x_1, \ldots, x_n) \rho f_A(y_1, \ldots, y_n)$$

(d) if $m \in \mathbb{N}$, $p \in \text{Pol}_A^m(P^*(A))$, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then

$$p(x_1, \ldots, x_m) \rho p(y_1, \ldots, y_m)$$

(e) if $m \in \mathbb{N}$, $t$ is an $m$-ary term, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then

$$t^P_A(x_1, \ldots, x_m) \rho t^P_A(y_1, \ldots, y_m)$$

(f) if $p \in \text{Pol}_A^1(P^*(A))$, $x, y \in A$ and $x \rho y$ then

$$p(x) \rho p(y)$$
Fundamental relation (generic)

(P., Purdea, 2006) For an equivalence relation \( \rho \) of a multialgebra \( A \), the following conditions are equivalent:

(a) \( A/\rho \) is a universal algebra (\( \rho \in E_{ua}(A) \));

(b) if \( n \in \mathbb{N} \), \( f \in \mathcal{F}_n \), \( a, b, x_1, \ldots, x_n \in A \), \( a \rho b \), then, for all \( i = 1, \ldots, n \),

\[
f^{A}(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \equiv f^{A}(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n);
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\[
f^A(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \overline{\rho} f^A(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n);
\]

(c) if $n \in \mathbb{N}$, $f \in F_n$, $x_i, y_i \in A$ and $x_i \rho y_i$ ($i = 1, \ldots, n$), then
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   \[ f^A(x_1, \ldots, x_n) \rho f^A(y_1, \ldots, y_n); \]

(d) if $m \in \mathbb{N}$, $p \in \text{Pol}^A_m(\mathbf{P}^*(A))$, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then
   \[ p(x_1, \ldots, x_m) \rho p(y_1, \ldots, y_m); \]
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(d) if $m \in \mathbb{N}$, $p \in \text{Pol}^A_m(P^*(A))$, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then
$$p(x_1, \ldots, x_m) \bar{\rho} p(y_1, \ldots, y_m);$$

(e) if $m \in \mathbb{N}$, $t$ is an $m$-ary term, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then
$$t^P^*(A)(x_1, \ldots, x_m) \bar{\rho} t^P^*(A)(y_1, \ldots, y_m).$$
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$$f^A(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \rho f^A(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n);$$

(c) if $n \in \mathbb{N}$, $f \in F_n$, $x_i, y_i \in A$ and $x_i \rho y_i$ ($i = 1, \ldots, n$), then

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(d) if $m \in \mathbb{N}$, $p \in \text{Pol}_m^A(P^*(A))$, $x_i, y_i \in A$, $x_i \rho y_i$ ($i = 1, \ldots, m$), then

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$$t^{P^*(A)}(x_1, \ldots, x_m) \rho t^{P^*(A)}(y_1, \ldots, y_m);$$

(f) if $p \in \text{Pol}_1^A(P^*(A))$, $x, y \in A$ and $x \rho y$ then

$$p(x) \rho p(y).$$
The $\mathcal{I}$-fundamental relation

(P., Purdea, 2006) $\alpha^*_q r = \text{the smallest relation from } E_{ua}(A) \text{ for which the factor (multi)algebra satisfies the identity } q = r$
The $\mathcal{I}$-fundamental relation

**Theorem (P., Purdea, Stanca)**

Let $\mathcal{I} = \{ q_i = r_i \mid i \in I \}$ ($I \neq \emptyset$, $q_i, r_i$ $m_i$-ary terms of type $\mathcal{F}$, $i \in I$) and let $A = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra. The smallest equivalence relation on $A$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I}$ (i.e. the $\mathcal{I}$-fundamental relation of $A$) is the transitive closure $\alpha^*_{\mathcal{I}}$ of the relation $\alpha_{\mathcal{I}} \subseteq A \times A$ defined by:

$$x \alpha_{\mathcal{I}} y \iff \exists i \in I, \exists p_i \in \text{Pol}_1^A(P^*(A)), \exists a_1^i, \ldots, a_{m_i}^i \in A :$$

$$x \in p_i^A(q_i^A(a_1^i, \ldots, a_{m_i}^i)), y \in p_i^A(r_i^A(a_1^i, \ldots, a_{m_i}^i)),$$

or

$$y \in p_i^A(q_i^A(a_1^i, \ldots, a_{m_i}^i)), x \in p_i^A(r_i^A(a_1^i, \ldots, a_{m_i}^i)).$$

(P., Purdea, 2006) $\alpha^*_{qr} = $ the smallest relation from $E_{ua}(A)$ for which the factor (multi)algebra satisfies the identity $q = r$

$$\implies \alpha^*_{\mathcal{I}} = \bigvee_{i \in I} \alpha^*_{q_i,r_i}.$$
The general recipe

For an $\mathcal{F}$-multialgebra $A = \langle A, F \rangle$, the poset $\langle E_{ua}(A), \subseteq \rangle$ is an algebraic closure system on $A \times A$. 

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The general recipe

For an $\mathcal{F}$-multialgebra $A = \langle A, F \rangle$, the poset $\langle E_{ua}(A), \subseteq \rangle$ is an algebraic closure system on $A \times A$. If $\alpha^A$ is the corresponding closure operator then

Theorem (Pelea, 2013)

If $\theta \subseteq A \times A$, the relation $\alpha^A(\theta)$ is defined as follows: $\langle x, y \rangle \in \alpha^A(\theta)$ if and only if there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$ of elements from $A$, some pairs $\langle b_1, c_1 \rangle, \ldots, \langle b_k, c_k \rangle \in \theta$ and some unary polynomial functions $p_1, \ldots, p_k$ from $\text{Pol}_1^A(\mathbf{P}^*(A))$ such that for all $i \in \{1, \ldots, k\}$,

$$\langle t_{i-1}, t_i \rangle \in p_i(b_i) \times p_i(c_i) \text{ or } \langle t_i, t_{i-1} \rangle \in p_i(b_i) \times p_i(c_i).$$
The general recipe

For an $\mathcal{F}$-multialgebra $A = \langle A, F \rangle$, the poset $\langle E_{ua}(A), \subseteq \rangle$ is an algebraic closure system on $A \times A$. If $\alpha^A$ is the corresponding closure operator then the $\mathcal{I}$-fundamental relation is $\alpha^A(R_{\mathcal{I}})$, where

$$R_{\mathcal{I}} = \bigcup \left\{ q_i^{P^*(A)}(a_1, \ldots, a_{m_i}) \times r_i^{P^*(A)}(a_1, \ldots, a_{m_i}) \mid a_1, \ldots, a_{m_i} \in A, i \in I \right\}$$

thus $\alpha^*_\mathcal{I}$ follows from the next result by taking $\theta = R_{\mathcal{I}}$.

**Theorem (Pelea, 2013)**

If $\theta \subseteq A \times A$, the relation $\alpha^A(\theta)$ is defined as follows: $\langle x, y \rangle \in \alpha^A(\theta)$ if and only if there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$ of elements from $A$, some pairs $\langle b_1, c_1 \rangle, \ldots, \langle b_k, c_k \rangle \in \theta$ and some unary polynomial functions $p_1, \ldots, p_k$ from $\text{Pol}_1^A(P^*(A))$ such that for all $i \in \{1, \ldots, k\}$,

$$\langle t_{i-1}, t_i \rangle \in p_i(b_i) \times p_i(c_i) \text{ or } \langle t_i, t_{i-1} \rangle \in p_i(b_i) \times p_i(c_i).$$
Applications... 

...to hypergroups

Let $\langle H, \cdot \rangle$ be a hypergroup.

$$E_{ua}(\langle H, \cdot \rangle) = \text{the set of the strongly regular equivalence relations}$$

The reproducibility condition defines the binary multioperations

$$/ \times \setcolon H^2 \to P^*(H), \quad b/a = \{x \in H \mid b \in x \cdot a\}, \quad a\setminus b = \{x \in H \mid b \in a \cdot x\},$$

but

$$E_{ua}(\langle H, \cdot \rangle) = E_{ua}(\langle H, \cdot, /, \setminus \rangle)$$

and $\langle H/\rho, \cdot \rangle$ is a group for any $\rho \in E_{ua}(\langle H, \cdot \rangle)$. 

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but

$$E_{ua}(\langle H, \cdot \rangle) = E_{ua}(\langle H, \cdot, /, \setminus \rangle)$$

and $\langle H/\rho, \cdot \rangle$ is a group for any $\rho \in E_{ua}(\langle H, \cdot \rangle)$.

(Koskas, 1970) The fundamental relation of $\langle H, \cdot \rangle$ is (the transitive closure $\beta^*$ of) the relation $\beta$ defined by

$$x\beta y \iff \exists n \in \mathbb{N}^*, \exists a_1, \ldots, a_n \in H : x, y \in a_1 \cdots a_n.$$
Applications... 

... to hypergroups and commutativity 

Let \( \langle H, \cdot \rangle \) be a hypergroup.

(Freni, 2002) the commutative-fundamental relation of \( \langle H, \cdot \rangle \) is (the transitive closure \( \gamma^* \) of) the relation \( \gamma \) defined by

\[
x \gamma y \iff \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},
\]

where

\[
\alpha \{ x_1 x_2 = x_2 x_1 \} = \bigcup_{n \in \mathbb{N}^*} \gamma_n,
\]

where

\[
\gamma_1 = \delta_H \quad \text{and for} \quad n > 1, \quad x \gamma_n y \iff \exists z_1, \ldots, z_n \in H, \exists i \in \{ 1, \ldots, n-1 \} : x \in z_1 \cdots z_i - 1 (z_i z_{i+1}) z_i + 2 \cdots z_n, \quad y \in z_1 \cdots z_i - 1 (z_i + 1 z_i) z_i + 2 \cdots z_n.
\]
Applications...

... to hypergroups and commutativity

Let $\langle H, \cdot \rangle$ be a hypergroup.

$$\alpha \{ x_1 x_2 = x_2 x_1 \} = \bigcup_{n \in \mathbb{N}^*} \gamma_n,$$

where $\gamma_1 = \delta_H$ and for $n > 1$,

$$x \gamma_n y \iff \exists z_1, \ldots, z_n \in H, \exists i \in \{1, \ldots, n-1\} :$$

$$x \in z_1 \cdots z_{i-1} (z_i z_{i+1}) z_{i+2} \cdots z_n, \ y \in z_1 \cdots z_{i-1} (z_{i+1} z_i) z_{i+2} \cdots z_n.$$
Applications...

... to hypergroups and commutativity

Let \( \langle H, \cdot \rangle \) be a hypergroup.

\[
\alpha\{x_1x_2=x_2x_1\} = \bigcup_{n \in \mathbb{N}^*} \gamma_n,
\]

where \( \gamma_1 = \delta_H \) and for \( n > 1 \),

\[
x \gamma_n y \iff \exists z_1, \ldots, z_n \in H, \exists i \in \{1, \ldots, n - 1\} : \\
x = z_1 \cdots z_{i-1}(z_iz_{i+1})z_{i+2} \cdots z_n, \ y = z_1 \cdots z_{i-1}(z_{i+1}z_i)z_{i+2} \cdots z_n.
\]

The transitive closure of \( \alpha\{x_1x_2=x_2x_1\} \) is

the commutative-fundamental relation of \( \langle H, \cdot \rangle \) is (the transitive closure \( \gamma^* \) of) the relation \( \gamma \) defined by

\[
x \gamma y \iff \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x = \prod_{i=1}^{n} z_i, \ y = \prod_{i=1}^{n} z_{\sigma(i)},
\]

since the cycles \((1 2),(2 3),\ldots,(n - 1 \ n)\) generate the symmetric group \( S_n \).
Applications... 

...to $n$-(semi)hypergroups 

Let $\langle H, f \rangle$ be an $n$-semihypergroup ($n \in \mathbb{N}, n \geq 2$).

(Davvaz, Fotea, 2008) the form of the fundamental relation of $\langle H, f \rangle$

(Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f \rangle$
Applications...  

... to n-(semi)hypergroups

Let $\langle H, f \rangle$ be an $n$-semihypergroup ($n \in \mathbb{N}$, $n \geq 2$).

Using the associativity of $f$ one obtains the form of the term functions from $\langle P^* (H), f \rangle$ and from the characterization of the fundamental relation of a (general) multialgebra one can immediately deduce (Davvaz, Fotea, 2008) the form of the fundamental relation of $\langle H, f \rangle$

If we take $I = \{ f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \mid \sigma \in S_n \}$, then $\alpha_I^*$ is (Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f \rangle$
Applications...  

...to $n$-(semi)hypergroups

Let $\langle H, f \rangle$ be an $n$-semihypergroup ($n \in \mathbb{N}$, $n \geq 2$).

Using the associativity of $f$ one obtains the form of the term functions from $\langle P^*(H), f \rangle$ and from the characterization of the fundamental relation of a (general) multialgebra one can immediately deduce (Davvaz, Fotea, 2008) the form of the fundamental relation of $\langle H, f \rangle$

If we take $I = \{ f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \mid \sigma \in S_n \}$, then $\alpha^*_{I}$ is (Davvaz, Dudek, Mirvakili, 2009) the commutative-fundamental relation of $\langle H, f \rangle$

- If $\langle H, f \rangle$ is $n$-hypergroup, it hides in its structure the multioperations

$$(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \mapsto \{x \mid b \in f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)\}$$

but in the corresponding fundamental algebra the associativity of $f$ and the existence of a solution for (each of) the equation $b = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ implies its uniqueness, thus

$$E_{ua}(\langle R, f \rangle) = E_{ua}(\langle R, f_1, \ldots, f_n \rangle).$$
Applications...

...to hyperrings

Let \( \langle A, +, \cdot \rangle \) be a hyperring.

(Vougiouklis, 1991) the fundamental relation \( \Gamma^* \) of \( \langle A, +, \cdot \rangle \) is the transitive closure of the relation consisting of all the pairs \( \langle x, y \rangle \) for which there exist \( n, k_1, \ldots, k_n \in \mathbb{N}^*, x_{i1}, \ldots, x_{ik_i} \in A \) \( (i = 1, \ldots, n) \) such that

\[
x, y \in \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{ij} \right).
\]
Applications... 

... to hyperrings

Let $\langle A, +, \cdot \rangle$ be a hyperring.

Since $\cdot$ is (sub)distributive with respect to $+$, any image of a term function of $\langle P^*(A), +, \cdot \rangle$ is contained in a sum of products of nonempty subsets of $A$, thus

the fundamental relation $\Gamma^*$ of $\langle A, +, \cdot \rangle$ is the transitive closure of the relation consisting of all the pairs $\langle x, y \rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, $x_{i1}, \ldots, x_{ik_i} \in A$ ($i = 1, \ldots, n$) such that

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Applications...

... to hyperrings and commutativity
Let \( \langle A, +, \cdot \rangle \) be a hyperring and \( I = \{ x_1 + x_2 = x_2 + x_1, \ x_1 \cdot x_2 = x_2 \cdot x_1 \} \).
Applications... 

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Let $\langle A, +, \cdot \rangle$ be a hyperring and $\mathcal{I} = \{x_1 + x_2 = x_2 + x_1, \ x_1 \cdot x_2 = x_2 \cdot x_1\}$.

(Davvaz, Vougiouklis, 2007) $\alpha^*_\mathcal{I}$ is the transitive closure $\alpha^*$ of the relation consisting of all the pairs $\langle x, y \rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, $\tau \in S_n$, $x_{i1}, \ldots, x_{ik_i} \in A$, and $\sigma_i \in S_{k_i} (i = 1, \ldots, n)$ such that

$$x \in \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{\tau(i)\sigma(i)(j)} \right).$$
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Let $\langle A, +, \cdot \rangle$ be a hyperring and $\mathcal{I} = \{x_1 + x_2 = x_2 + x_1, \ x_1 \cdot x_2 = x_2 \cdot x_1\}$.

(Davvaz, Vougiouklis, 2007) $\alpha^*_I$ is the transitive closure $\alpha^*$ of the relation consisting of all the pairs $\langle x, y \rangle$ for which there exist $n, k_1, \ldots, k_n \in \mathbb{N}^*$, $\tau \in S_n$, $x_{i1}, \ldots, x_{ik_i} \in A$, and $\sigma_i \in S_{k_i}(i = 1, \ldots, n)$ such that

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(P., Purdea, Stanca) $\alpha^*_I$ is the transitive closure of the union of all the Cartesian products

$$t(a_1, \ldots, a_m) \times t'(a_1, \ldots, a_m),$$
where $t(a_1, \ldots, a_m)$ is a sum of products of elements of $A$ (we allow the sum to have only one term and the products to have only one factor), and $t'(a_1, \ldots, a_m)$ is obtained from $t(a_1, \ldots, a_m)$ either by permuting two consecutive factors in a product or by permuting two consecutive terms in the sum.
Moreover...

Theorem (P., Purdea, Stanca)

The variety $M(I)$ of the $F$-algebras which satisfy all the identities from $I$ is a reflective subcategory of the category of $F$-multialgebras.
Moreover...

Theorem (P., Purdea, Stanca)

The variety \( M(\mathcal{I}) \) of the \( \mathcal{F} \)-algebras which satisfy all the identities from \( \mathcal{I} \) is a reflective subcategory of the category of \( \mathcal{F} \)-multialgebras.

Some corollaries...
Moreover...

**Theorem (P., Purdea, Stanca)**

The variety $M(\mathcal{I})$ of the $\mathcal{F}$-algebras which satisfy all the identities from $\mathcal{I}$ is a reflective subcategory of the category of $\mathcal{F}$-multialgebras.

**Some corollaries . . .**

- The factorization of $\mathcal{F}$-multialgebras modulo $\alpha^*_\mathcal{I}$ provides a functor $F_\mathcal{I}$ which is a reflector for $M(\mathcal{I})$: 

\[
F_\mathcal{I}(A) = A/\alpha^*_\mathcal{I}
\]
Moreover...

Theorem (P., Purdea, Stanca)

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Some corollaries...

• The factorization of $F$-multialgebras modulo $\alpha^*_I$ provides a functor $F_I$ which is a reflector for $M(I)$:
  
  $F_I$ is defined by $F_I(A) = A/\alpha^*_I$ on objects,
Moreover... 

Theorem (P., Purdea, Stanca)

The variety $M(\mathcal{I})$ of the $\mathcal{F}$-algebras which satisfy all the identities from $\mathcal{I}$ is a reflective subcategory of the category of $\mathcal{F}$-multialgebras.

Some corollaries...

- The factorization of $\mathcal{F}$-multialgebras modulo $\alpha_\mathcal{I}^*$ provides a functor $F_\mathcal{I}$ which is a reflector for $M(\mathcal{I})$:
  - $F_\mathcal{I}$ is defined by $F_\mathcal{I}(A) = A/\alpha_\mathcal{I}^*$ on objects,
  - $F_\mathcal{I}$ assigns to each multialgebra homomorphism $h : A \to B$ the unique universal algebra homomorphism $\overline{h}$ which makes the diagram
    \[
    \begin{array}{ccc}
    A & \xrightarrow{h} & B \\
    \downarrow{\varphi_A} & & \downarrow{\varphi_B} \\
    A/\alpha_\mathcal{I}^* & \xrightarrow{\overline{h}} & B/\alpha_\mathcal{I}^*
    \end{array}
    \]
    commutative (the vertical arrows are the canonical projections).
Other corollaries...

Let $\mathcal{I}$ be a set of identities and let $A = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra.
**Other corollaries**

Let $\mathcal{I}$ be a set of identities and let $A = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra.

- $F_\mathcal{I}$ preserves the directed colimits.

For a hyperring $\langle A, +, \cdot \rangle$ with $+$ weak commutative or with a multiplicative identity, its fundamental relation is $\alpha^* \{ x_1 + x_2 = x_2 + x_1 \}$, $\alpha^* = \alpha^* \{ x_1 x_2 = x_2 x_1 \}$ and the ring $A/\alpha^*$ is isomorphic to the factor of the fundamental ring of $A$ over its commutator ideal.
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Other corollaries . . .

Let $I$ be a set of identities and let $A = \langle A, F \rangle$ be an $F$-multialgebra.

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- $A/\alpha^*_I \cong (A/\alpha^*_A)/\alpha^*_I$.  

The commutative-fundamental group of a hypergroup is the abelianization of its fundamental group.

For a hyperring $\langle A, +, \cdot \rangle$ with $+$ weak commutative or with a multiplicative identity, its fundamental relation is $\alpha^*_\ast \{x_1 + x_2 = x_2 + x_1\}$, $\alpha^*_\ast = \alpha^*_\ast \{x_1 x_2 = x_2 x_1\}$ and the ring $A/\alpha^*_I$ is isomorphic to the factor of the fundamental ring of $A$ over its commutator ideal.
Other corollaries . . .

Let $\mathcal{I}$ be a set of identities and let $A = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra.

- $F_{\mathcal{I}}$ preserves the directed colimits.

(One should notice that, as in the case of the “fundamental functor” $F = F_{\{x=x\}}$, in general $F_{\mathcal{I}}$ does not preserve the (finite) products.)

- $A/\alpha^*_\mathcal{I} \simeq (A/\alpha^*_A)/\alpha^*_\mathcal{I}$.

- The commutative-fundamental group of a hypergroup is the abelianization of its fundamental group.
Other corollaries . . .

Let $\mathcal{I}$ be a set of identities and let $\mathbf{A} = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra.

- $F_\mathcal{I}$ preserves the directed colimits.

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- $\mathbf{A}/\alpha^*_\mathcal{I} \cong (\mathbf{A}/\alpha^*_\mathbf{A})/\alpha^*_\mathcal{I}$.

- The commutative-fundamental group of a hypergroup is the abelianization of its fundamental group.

- For a hyperring $\langle A, +, \cdot \rangle$ with $+$ weak commutative or with a multiplicative identity, its fundamental relation is $\alpha^*_\{x_1 + x_2 = x_2 + x_1\}$,

$$\alpha^* = \alpha^*_\{x_1 x_2 = x_2 x_1\}$$

and the ring $A/\alpha^*$ is isomorphic to the factor of the fundamental ring of $A$ over its commutator ideal.
... and other applications ...

(Ameri, Nozari, 2010) the fundamental relation of a fuzzy hyperalgebra
... and other applications . . .

(Ameri, Nozari, 2010) the fundamental relation of a fuzzy hyperalgebra

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(P., Purdea, Stanca) The category of $\mathcal{F}$-multialgebras is isomorphic to a reflective subcategory of the category of $\mathcal{F}$-fuzzy hyperalgebras (with fuzzy hyperalgebra homomorphisms and mapping composition).

\[ \implies \] any variety of $\mathcal{F}$-algebras is isomorphic to a reflective subcategory of the category of $\mathcal{F}$-fuzzy hyperalgebras.

\[ \implies \] the composition of the isomorphism resulting from the proof with the corresponding reflector provides a functor $G$ from the category of $\mathcal{F}$-fuzzy hyperalgebras into the category of $\mathcal{F}$-multialgebras.
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$\implies$ the fundamental algebra of a fuzzy hyperalgebra $A$ is the algebra
$(F \circ G)(A)$ ($F = F_{x=x}$)
... and other applications . . .

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(P., Purdea, Stanca) The category of $\mathcal{F}$-multialgebras is isomorphic to a reflective subcategory of the category of $\mathcal{F}$-fuzzy hyperalgebras (with fuzzy hyperalgebra homomorphisms and mapping composition).

$\Rightarrow$ any variety of $\mathcal{F}$-algebras is isomorphic to a reflective subcategory of the category of $\mathcal{F}$-fuzzy hyperalgebras.

$\Rightarrow$ the composition of the isomorphism resulting from the proof with the corresponding reflector provides a functor $G$ from the category of $\mathcal{F}$-fuzzy hyperalgebras into the category of $\mathcal{F}$-multialgebras.

$\Rightarrow$ the fundamental algebra of a fuzzy hyperalgebra $A$ is the algebra $(F \circ G)(A) (F = F_{\{x=x\}})$

- Replacing $F$ by $F_{\mathcal{I}}$ in the above statement opens up the possibility to define the $\mathcal{I}$-fundamental algebra of a fuzzy multialgebra for any set $\mathcal{I}$ of identities.
and other applications

(P., Purdea, Stanca) If $A$ is a multialgebra, $\mu : A \rightarrow [0, 1]$ is a fuzzy submultialgebra of $A$ and $\rho \in E_{ua}(A)$, then $\mu_{\rho}$ defined by

$$\mu_{\rho}(y/\rho) = \bigvee \{\mu(x) \mid \langle x, y \rangle \in \rho\}$$

is a fuzzy subalgebra of $A/\rho$. 
and other applications

(P., Purdea, Stanca) If $A$ is a multialgebra, $\mu : A \to [0, 1]$ is a fuzzy submultialgebra of $A$ and $\rho \in E_{ua}(A)$, then $\mu_\rho$ defined by

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$\implies \qquad \mu_{\alpha^*_I}$ is a fuzzy subalgebra of $A/\alpha^*_I$. 
(P., Purdea, Stanca) If $A$ is a multialgebra, $\mu : A \to [0, 1]$ is a fuzzy submultialgebra of $A$ and $\rho \in E_{ua}(A)$, then $\mu_{\rho}$ defined by

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is a fuzzy subalgebra of $A/\rho$.

$\implies \mu_{\alpha^*_I}$ is a fuzzy subalgebra of $A/\alpha^*_I$.

$\implies \mu_{\alpha^*_A}$ is a fuzzy subalgebra of the fundamental algebra of $A$. 
For details, see ...


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