Multialgebras, factor multialgebras and universal algebras

Cosmin Pelea

(Based on a joint work with Ioan Purdea)

A recipe:

(F. Marty, 8th Congress of the Scandinavian Mathematicians, Stockholm, 1934)

- Let $(G, \cdot)$ be a group, $H \leq G$ and $G/H = \{xH \mid x \in G\}$. The equality

$$(1) \qquad (xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

defines an operation on G/H if and only if $H \trianglelefteq G$.

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  defines an operation on G/H if and only if $H \trianglelefteq G$.
- In general, (1) defines a function

  $$G/H \times G/H \to P^*(G/H)$$

  called binary multioperation (on G/H) (and $(G/H,\cdot)$ is a hypergroup).

Multialgebra

An n-ary multioperation f on a set A is a mapping

$$f : A^n \to P^*(A).$$

($P^*(A)$ denotes the set of the nonempty subsets of A).

Let $\mathcal{F}$ be a type of (multi)algebras.

A multialgebra $\mathbf{A} = (A, F)$ of type $\mathcal{F}$ consists of a set A and a family of multioperations F obtained by associating a multioperation $f^{\mathbf{A}}$ (or, simply, f) on A to each symbol f from $\mathcal{F}$.

The recipe:

Later, in 'A representation theorem for multi-algebras', *Arch. Math.*, **3** 1962, 452–456, G. Grätzer proved that:

Any multialgebra **A** *results from a universal algebra* **B** *and an appropriate equivalence on* B *as before, i.e. by taking*

$$f(a_1/\rho, \ldots, a_n/\rho) = \{b/\rho \mid b = f(b_1, \ldots, b_n),\ a_i \rho b_i,\ i \in \{1, \ldots, n\}\}.$$

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$\Rightarrow$ the importance of the factor multialgebras in multialgebra theory.

The start:

M. Dresher, O. Ore, 'Theory of multigroups', *Amer. J. Math.*, **60** 1938, 705–733.

M. Koskas, 'Groupoïdes, demi-hypergroupes et hypergroupes', *J. Math. Pures et Appl.*, **49**, 1970, 155–192.

D. Freni, 'A new characterization of the derived hypergroup via strongly regular equivalences', *Comm. Algebra*, **30** 2002, 3977–3989.

$\Rightarrow$ the importance of the equivalence relations of (semi)hypergroups for which the factor hypergroup(oid) is a group (strongly regular equivalences)

The start:

Let $(H, \cdot)$ be a (semi)hypergroup.

• the smallest strongly regular equivalence of $(H, \cdot)$ = the fundamental relation of $(H, \cdot)$ = the transitive closure β^* of

$$x \beta y \Leftrightarrow \exists n \in \mathbb{N}^*,\ \exists a_1, \ldots, a_n \in H:\ x, y \in a_1 \cdots a_n$$

(if $(H, \cdot)$ is a hypergroup, then $\beta^* = \beta$)

• the smallest strongly regular equivalence of $(H, \cdot)$ for which the factor hypergroup is a commutative group = the transitive closure γ^* of the relation

$$x \gamma y \Leftrightarrow \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i,\ y \in \prod_{i=1}^{n} z_{\sigma(i)}$$

(if $(H, \cdot)$ is a hypergroup, then $\gamma^* = \gamma$)

The problems:

1. Characterize the fundamental relation α^* for a (general) multialgebra and prove that

$$(\mathbf{B}/\rho)/\alpha^* \cong \mathbf{B}/\theta(\rho),$$

($\mathbf{B}$ is a universal algebra, ρ is an equivalence relation on B, and $\theta(\rho)$ is the smallest congruence relation on $\mathbf{B}$ which contains ρ)

The problems:

2. Let q, r be n-ary terms of type $\mathcal{F}$. Determine the smallest equivalence relation α^*_{qr} of a (general) multialgebra for which the factor multialgebra is a universal algebra satisfying the identity

$$q = r.$$

If $\mathbf{B}$ is a universal algebra, ρ is an equivalence on B, and $\theta(\rho_{qr})$ is the congruence of $\mathbf{B}$ generated by

$$\rho \cup \{(q(b_1, \dots, b_n), r(b_1, \dots, b_n)),\ b_1, \dots, b_n \in B\}),$$

prove that

$$(\mathbf{B}/\rho)/\alpha^*_{qr} \cong \mathbf{B}/\theta(\rho_{qr}),$$

The tools:

For a multialgebra $\mathbf{A} = (A, F)$ we consider:

- the universal algebra $\mathbf{P}^*(\mathbf{A})$ of the nonempty subsets of $\mathbf{A}$, defined by

$$f^{\mathbf{P}^*(\mathbf{A})}(A_1, \ldots, A_n) = \bigcup \{ f^{\mathbf{A}}(a_1, \ldots, a_n) \mid \ a_i \in A_i, \ i = 1, \ldots, n \}$$

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- the subalgebra $\mathbf{P}^{(n)}_{A}(\mathbf{P}^*(\mathbf{A}))$ of $\mathbf{P}^{(n)}_{P^*(A)}(\mathbf{P}^*(\mathbf{A}))$ generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{1, \ldots, n\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are given by

$$c^n_a(A_1, \ldots, A_n) = \{a\} \text{ and } e^n_i(A_1, \ldots, A_n) = A_i;$$

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$$c^n_a(A_1, \ldots, A_n) = \{a\} \text{ and } e^n_i(A_1, \ldots, A_n) = A_i;$$
- the algebra $\mathbf{P}^{(n)}(\mathbf{P}^*(\mathbf{A}))$ of the n-ary term functions of $\mathbf{P}^*(\mathbf{A})$;
- the set $E_{ua}(\mathbf{A})$ of the equivalence relations ρ on A for which $\mathbf{A}/\rho$ is a universal algebra (which is an algebraic closure system on $A \times A$).

The results:

- the smallest equivalence relation α_{qr}^* from $E_{ua}(\mathbf{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $q = r$ is the transitive closure of the relation

$$\begin{aligned} x \alpha_{qr} y \ \Leftrightarrow\ & \exists p \in P_A^{(1)}(\mathbf{P}^*(\mathbf{A})),\ \exists\ a_1, \ldots, a_n \in A : \\ & x \in p(q(a_1, \ldots, a_n)),\ y \in p(r(a_1, \ldots, a_n)) \text{ or} \\ & y \in p(q(a_1, \ldots, a_n)),\ x \in p(r(a_1, \ldots, a_n)); \end{aligned}$$

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- the correspondence

$$(B/\rho)/\alpha_{qr}^* \to B/\theta(\rho_{qr}),\ (a/\rho)/\alpha_{qr}^* \mapsto a/\theta(\rho_{qr})$$

is a universal algebra isomorphism.

The results:

Taking q and r to be the same variable, we have:

- the fundamental relation α^* of the multialgebra $\mathbf{A}$ (i.e. the smallest element from $E_{ua}(\mathbf{A})$) is the transitive closure of the relation

$$x \alpha y \Leftrightarrow \exists p \in P_A^{(1)}(\mathbf{P}^*(\mathbf{A})), \exists a \in A : x, y \in p(a);$$

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$$x\alpha y \Leftrightarrow x, y \in p(a_1, \ldots, a_n)$$

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for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathbf{P}^*(\mathbf{A}))$ and $a_1, \dots, a_n \in A$;

- the correspondence $(a/\rho)/\alpha^* \mapsto a/\theta(\rho)$ provides the universal algebra isomorphism

$$(\mathbf{B}/\rho)/\alpha^* \to \mathbf{B}/\theta(\rho).$$

A return to hypergroups:

Let $(H, \cdot)$ be a hypergroup.

$$\alpha_{x_1x_2,x_2x_1} = \bigcup_{n\in\mathbb{N}^*} \gamma_n,$$

where $\gamma_1 = \delta_H$ and for $n > 1$,

$$x\gamma_n y \Leftrightarrow \exists z_1, \ldots, z_n \in H,\ \exists i \in \{1, \ldots, n-1\} : \\ x \in z_1 \cdots z_{i-1}(z_i z_{i+1})z_{i+2} \cdots z_n, \\ y \in z_1 \cdots z_{i-1}(z_{i+1} z_i)z_{i+2} \cdots z_n,$$

The strongly regular relation γ^* is the transitive closure of $\alpha_{x_1x_2,x_2x_1}$ which is, clearly, (the transitive closure of) the relation

$$x\gamma y \Leftrightarrow \exists n \in \mathbb{N}^*, \exists z_1, \ldots, z_n \in H, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

since the cycles $(1,2), (2,3), \ldots, (n-1,n)$ generate the symmetric group $(S_n, \circ)$.

A return to hypergroups:

The derived subhypergroup of a hypergroup $(K, \cdot)$ is

$$D(K) = \varphi_K^{-1}(1_{K/\gamma}),$$

where $\varphi_K : K \to K/\gamma$ is the canonical projection and $1_{K/\gamma}$ is the identity of the group $(K/\gamma, \cdot)$.

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Let $(G, \cdot)$ be a group, G' its derived subgroup, $H \leq G$ and $(G/H, \cdot)$ the hypergroup defined by

$$(1) \qquad (xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

Let $\pi_H : G \to G/H$ and $\varphi_{G/H} : G/H \to (G/H)/\gamma$ be the canonical projections. The group isomorphism

$$h : (G/H)/\gamma \to G/(G'H),\ h((xH)/\gamma) = x(G'H)$$

helps us provide the following connection between the derived subhypergroup of G/H and the derived subgroup of G

$$D(G/H) = (h \circ \varphi_{G/H})^{-1}(G'H) = (G'H)/H = \pi_H(G').$$

For details, see...

Pelea, C.; Purdea, I.: *Multialgebras, universal algebras and identities*, J. Aust. Math. Soc, **81**, 2006, 121–139.

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