Identities in multialgebra theory

Cosmin Pelea

Keywords and tools

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Multialgebra

Let $\tau = (n_\gamma)_{\gamma < o(\tau)}$ be a sequence of nonnegative integers ($o(\tau)$ is an ordinal) and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation. Denote by

$$\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma < o(\tau)})$$

the algebra of the n-ary terms (of type τ).

A *multialgebra* $\mathfrak{A}$ *of type* τ consists in a set A and a family of multioperations $(f_\gamma)_{\gamma < o(\tau)}$, where

$$f_\gamma : A^{n_\gamma} \to P^*(A)$$

is the n_γ-ary multioperation which corresponds to the symbol $\mathbf{f}_\gamma$.

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$$(a_0, \ldots, a_{n_\gamma - 1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow \ a_{n_\gamma} \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1}).$$

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  A multialgebra $\mathfrak{A}$ determines a universal algebra $\mathfrak{P}^*(A)$ on $P^*(A)$ defining for any $A_0, \ldots, A_{n_\gamma - 1} \in P^*(A)$,

$$f_\gamma(A_0, ..., A_{n_\gamma - 1}) = \bigcup \{ f_\gamma(a_0, ..., a_{n_\gamma - 1}) \mid a_i \in A_i, i = 0, ..., n_\gamma - 1 \}.$$

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$$\mathbf{q} = \mathbf{r}$$

is satisfied on a multialgebra $\mathfrak{A}$ if

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(q and r denotes the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ on $\mathfrak{P}^*(A)$.)

Let $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ be the algebra of the n-ary polynomial functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$ and $\mathfrak{P}^{(n)}_{A}(\mathfrak{P}^*(\mathfrak{A}))$ its subalgebra generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \ldots, n-1\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are defined by

$$c^n_a(A_0, \ldots, A_{n-1}) = \{a\} \text{ and } e^n_i(A_0, \ldots, A_{n-1}) = A_i.$$

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The algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$ is the subalgebra of $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ generated by

$$\{e^n_i \mid i \in \{0, \ldots, n-1\}\}.$$

Factor multialgebra

Let ρ be an equivalence relation on A, let $\rho\langle x\rangle$ be the class of x modulo ρ, and

$$A/\rho = \{\rho\langle x\rangle \mid x \in A\}.$$

Defining for each $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma - 1}\rangle) = \{\rho\langle b\rangle | b \in f_\gamma(b_0, ..., b_{n_\gamma - 1}), a_i \rho b_i, i = \overline{0, n_\gamma - 1}\},$$

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- G. Grätzer proved that *any multialgebra is a factor of a universal algebra modulo an equivalence relation.*

Direct product of multialgebras

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. The Cartesian product $\prod_{i\in I} A_i$ with the multioperations

$$f_\gamma((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma - 1})_{i\in I}) = \prod_{i\in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}),$$

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- If $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$ then

$$p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1}).$$

Direct limit of a direct system of multialgebras

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let A_∞ be the direct limit of the direct system of their supporting sets.

Let us remind that:

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Let us remind that:

- $(I, \leq)$ is a directed preordered set;
- the set A_∞ is the factor of the disjoint union A of the sets A_i modulo the equivalence relation $\equiv$ defined as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and

$$x \equiv y \ \Leftrightarrow\ \exists k \in I,\ i \leq k,\ j \leq k :\ \varphi_{ik}(x) = \varphi_{jk}(y).$$

We define the multioperations f_γ on $A_\infty = \{\widehat{x} \mid x \in A\}$ as follows: if $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}} \in A_\infty$ and for any $j \in \{0, ..., n_\gamma - 1\}$ we consider that $x_j \in A_{i_j}$ $(i_j \in I)$ then

$$f_\gamma(\widehat{a_0}, \ldots, \widehat{a_{n_\gamma-1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I, i_0 \leq m, \ldots, i_{n_\gamma-1} \leq m, \\ a \in f_\gamma(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n_\gamma-1} m}(a_{n_\gamma-1}))\}.$$

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- If $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$, $a_0, \ldots, a_{n-1} \in A$ and $i_0, \ldots, i_{n-1} \in I$ are such that $a_j \in A_{i_j}$ for all $j \in \{0, \ldots, n-1\}$ then

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2. *How identities acts with respect to some constructions of multialgebras?*

(Some) ANSWERS

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A semihypergroup $(H, \circ)$ is a multialgebra with one binary multioperation satisfying the identity

$$(1) \qquad (\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 = \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2).$$

The H_v-semihypergroups are obtained the same way, but, instead of (1) we have

$$(1') \qquad (\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 \cap \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \neq \emptyset.$$

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YES opens the possibility to characterize some *hypergroups* (for example, *canonical hypergroups*), some *hyperrings*, *H_v-rings*, *hypermodules* ... using identities.

Hypergroups and identities

Let $H \neq \emptyset$. A hypergroup $(H, \circ)$ is a semihypergroup which satisfies the condition:

$$a \circ H = H \circ a = H, \ \forall a \in H.$$

$\Rightarrow$ the maps $/, \backslash : H \times H \to P^*(H)$ defined by

$$(2) \qquad b/a = \{x \in H \mid b \in x \circ a\}, \ a \backslash b = \{x \in H \mid b \in a \circ x\},$$

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- $\Rightarrow$ a hypergroup $(H, \circ)$ can be seen as a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations, with $H \neq \emptyset$, which satisfy (1) and (2).
- $\Rightarrow$ a hypergroup is a nonempty multialgebra $(H, \circ, /, \backslash)$ of type $(2, 2, 2)$ which satisfy (1) and

$$(3) \qquad \mathbf{x}_1 \cap (\mathbf{x}_1/\mathbf{x}_0) \circ \mathbf{x}_0 \neq \emptyset, \ \mathbf{x}_1 \cap \mathbf{x}_0 \circ (\mathbf{x}_0 \backslash \mathbf{x}_1) \neq \emptyset.$$

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Example

If $(H, \circ)$ is a group with $|H| \geq 2$, and the multioperations $/, \backslash : H \times H \to P^*(H)$ are defined by

$$a/b = a\backslash b = H, \ \forall a, b \in H$$

then $(H, \circ)$ is a hypergroup, $(H, \circ, /, \backslash)$ satisfies the identities (3), but the sets

$$\{x \in H \mid a \in x \circ b\} = \{x \in H \mid a = x \circ b\},$$
$$\{x \in H \mid a \in b \circ x\} = \{x \in H \mid a = b \circ x\}$$

are singletons for any $a, b \in H$, so, they cannot be H.

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- *Direct limits of direct systems of multialgebras*

Factor multialgebras of universal algebras

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- *What happens with an identity of an universal algebra when we factorize it modulo an equivalence relation?*
- **Answer:** They become weak identities in the multialgebra we obtained this way.

$\Rightarrow$ the factor of a group (ring) is an H_v-group (H_v-ring)

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- The (strong) associativity of a binary operation is not satisfied in the factor multialgebra.
- The identities which characterize the commutativity of an operation of a universal algebra hold strongly on the factor multialgebra.
- The factor of a group modulo an equivalence relation is an H_v-group with identity and each element of this H_v-group has (at least) an inverse.

Factor multialgebras which are universal algebras

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra and ρ an equivalence relation on A. Remember that for $X, Y \in P^*(A)$,

$$X \overline{\overline{\rho}} Y \Leftrightarrow x\rho y,\ \forall x \in X,\ \forall y \in Y \Leftrightarrow X \times Y \subseteq \rho.$$

Proposition

The following conditions are equivalent:

a) $\mathfrak{A}/\rho$ *is a universal algebra;*

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a) *$\mathfrak{A}/\rho$ is a universal algebra;*

b) *If $\gamma < o(\tau)$, $a, b, x_i \in A$, $i \in \{0, \ldots, n_\gamma - 1\}$, with $a\rho b$, then*

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma - 1}) \overline{\overline{\rho}} f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma - 1})$$

for all $i \in \{0, \ldots, n_\gamma - 1\}$;

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a) $\mathfrak{A}/\rho$ *is a universal algebra;*

b) If $\gamma < o(\tau)$, $a, b, x_i \in A$, $i \in \{0, \dots, n_\gamma - 1\}$, *with* $a\rho b$, *then*

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma-1})\overline{\overline{\rho}}f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma-1})$$

for all $i \in \{0, \dots, n_\gamma - 1\}$*;*

c) If $\gamma < o(\tau)$, $x_i, y_i \in A$ *and* $x_i\rho y_i$ *for any* $i \in \{0, ..., n_\gamma - 1\}$, *then*

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Factor multialgebras which are universal algebras

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra and ρ an equivalence relation on A. Remember that for $X, Y \in P^*(A)$,

$$X\overline{\overline{\rho}}Y \;\Leftrightarrow\; x\rho y,\ \forall x \in X,\ \forall y \in Y \;\Leftrightarrow\; X \times Y \subseteq \rho.$$

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d) If $n \in \mathbb{N}$, $\mathbf{p} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ *and* $x_i, y_i \in A$ *with* $x_i \rho y_i$ *for any* $i \in \{0, \ldots, n - 1\}$, *then*

$$p(x_0, \ldots, x_{n-1})\overline{\overline{\rho}}p(y_0, \ldots, y_{n-1}).$$

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$\Rightarrow$ any identity (weak or strong) which holds on $\mathfrak{A}$ is also satisfied in its fundamental algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*$.

- Even if $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is not satisfied on $\mathfrak{A}$ we can obtain a factor multialgebra of $\mathfrak{A}$ which is a universal algebra satisfying $\mathbf{q} = \mathbf{r}$ by taking the factor multialgebra determined by a relation from $E_{ua}(\mathfrak{A})$ which contains

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- In particular, $\alpha^* = \alpha^*_{\mathbf{x}_0\mathbf{x}_0}$.

- α^* is the transitive closure of the relation α defined by

$$
\begin{aligned}
x\alpha y \;\Leftrightarrow\;& \exists n \in \mathbb{N},\ \exists p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists a_0, \ldots, a_{n-1} \in A: \\
& \qquad x, y \in p(a_0, \ldots, a_{n-1}) \\
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\begin{aligned}
x\alpha_{\mathbf{qr}} y \ \Leftrightarrow\ & \exists p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists\ a_0, \ldots, a_{n-1} \in A : \\
& x \in p(q(a_0, \ldots, a_{n-1})),\ y \in p(r(a_0, \ldots, a_{n-1})) \text{ or} \\
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- for (semi)hypergroups, $\alpha^*_{\mathbf{x}_0\mathbf{x}_1,\mathbf{x}_1\mathbf{x}_0}=\gamma^*$ introduced by D. Freni to characterize the derived subhypergroup of a hypergroup.

Back to factor of universal algebras

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ $(n \in \mathbb{N})$, let $\mathfrak{B}$ be a universal algebra and ρ an equivalence relation on B. We denote by $\rho_{\mathbf{qr}}$ the smallest equivalence relation on B containing ρ and

$$\{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}.$$

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- So,
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The details

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