ON THE FUNDAMENTAL RELATION OF A MULTIALGEBRA

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Abstract. The object of this paper are multialgebras. The purpose of this paper is to give the form of the fundamental relation of a multialgebra, i.e. the smallest equivalence relation for which the quotient set, considered as a multialgebra, is a universal algebra.

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence over $\mathbb{N} = \{0, 1, ...\}$, where $o(\tau)$ is an ordinal. Let A be a nonvoid set and $P^*(A)$ the family of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where $f_\gamma : A^{n_\gamma} \to P^*(A)$ is a multioperation of arity $n_\gamma \in \mathbb{N}$, for any $\gamma < o(\tau)$. $\mathfrak{A}$ induces a universal algebra $(P^*(A), (f_\gamma)_{\gamma<o(\tau)})$ with the operations:

$$f_\gamma(A_0, ..., A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, ..., a_{n_\gamma-1}) \mid \ a_i \in A_i, \ \forall i \in \{0, ..., n_\gamma - 1\}\},$$

for any $\gamma < o(\tau)$ and $A_0, ..., A_{n_\gamma-1} \in P^*(A)$. We denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$.

In [3] Grätzer presents the algebra of the term functions of a universal algebra $\mathfrak{B} = (B, (f_\gamma)_{\gamma<o(\tau)})$. If we add to the set of the operations of $\mathfrak{B}$ the nullary operations corresponding to the elements of B (i.e. the functions $\{\emptyset\} \to B$, $\emptyset \mapsto b$, for all $b \in B$), the n-ary term functions of this new algebra are called the n-ary polynomial functions of $\mathfrak{B}$. The n-ary polynomial functions $P^{(n)}(\mathfrak{B})$ of $\mathfrak{B}$ form a universal algebra with the operations $(f_\gamma)_{\gamma<o(\tau)}$, denoted by $\mathfrak{P}^{(n)}(\mathfrak{B})$.

For any $n \in \mathbb{N}$, we can construct the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of n-ary polynomial functions on $\mathfrak{P}^*(\mathfrak{A})$. Consider the subalgebra $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ of $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ obtained by adding to the operations $(f_\gamma)_{\gamma<o(\tau)}$ of $\mathfrak{P}^*(\mathfrak{A})$ only the nullary operations associated to the elements of A (we identify $\{a\}$ with a).

Thus the elements of $P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ $(n \in \mathbb{N})$ are those and only those functions from $(P^*(A))^n$ into $P^*(A)$ which can be obtained by applying (i), (ii) and (iii) from bellow for finitely many times:

(i) the functions $c^n_a : (P^*(A))^n \to (P^*(A))$ defined by setting $c^n_a(X_0, ..., X_{n-1}) = a$, for all $X_0, ..., X_{n-1} \in P^*(A)$ are elements of $P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$, for every $a \in A$;

(ii) the functions $e^n_i : (P^*(A))^n \to P^*(A)$, $e^n_i(X_0, ..., X_{n-1}) = X_i$, for all $X_0, ..., X_{n-1} \in P^*(A)$, $i = 0, ..., n-1$ are elements of $P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$;

(iii) if $p_0, ..., p_{n_\gamma - 1}$ are elements of $P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $\gamma < o(\tau)$ then $f_\gamma(p_0, ..., p_{n_\gamma-1}) : (P^*(A))^n \to P^*(A)$ defined by setting for all $X_0, ..., X_{n-1} \in P^*(A)$,

$$(f_\gamma(p_0, ..., p_{n_\gamma-1}))(X_0, ..., X_{n-1}) = f_\gamma(p_0(X_0, ..., X_{n-1}), ..., p_{n_\gamma-1}(X_0, ..., X_{n-1}))$$

is also an element of $P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$.

In this paper we will use only polynomial functions from $P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ thus we will drop the subscript A with no danger of confusion.

1. Definition. Let α be the relation defined on A as follows: for $x, y \in A$ set $x\alpha y$ if $x, y \in p(a_0, ..., a_n)$ for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$, and $a_0, ..., a_n \in A$.

It is clear that α is symmetric. Because any $a \in A$ is an element of $e_0^1(a)$, the relation α is also reflexive. We take α^* to be the transitive closure of α. Then α^* is an equivalence relation on A.

Recall that for a given equivalence relation ρ on A, for $X, Y \subseteq A$, we write $X \mathrel{\overline{\overline{\rho}}} Y$ if and only if $x\rho y$ holds for all $x \in X$, $y \in Y$.

2. Lemma. *If $f \in P^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a, b \in A$ satisfy $a\ \alpha^*\ b$ then $f(a) \mathrel{\overline{\overline{\alpha^*}}} f(b)$.*

Proof. By the definition of α^*,

$$a = y_0\ \alpha\ y_1\ \alpha\ ...\ \alpha\ y_{m-1} = b$$

for some $m \in \mathbb{N}$ and $y_1, ..., y_{m-2} \in A$. Let $u_i \in f(y_i)$ be arbitrary $(i = 0, ..., m-1)$. Consider $0 \le j < m-1$. Clearly, $y_j\ \alpha\ y_{j+1}$ means that $y_j, y_{j+1} \in p_j(a_0, ..., a_{n-1})$ for some $n_j \in \mathbb{N}$, $p_j \in P^{(n_j)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, ..., a_{n_j-1} \in A$. Define the n_j-ary multioperation q on A by setting

$$q_j(x_0, ..., x_{n_j-1}) = f(p_j(x_0, ..., x_{n_j-1}))$$

for all $x_0, ..., x_{n_j-1} \in A$. Clearly, $q_j \in P^{(n_j)}(\mathfrak{P}^*(\mathfrak{A}))$ and

$$q_j(a_0, ..., a_{n_j-1}) = \bigcup\{f(z)|\ z \in p_j(a_0, ..., a_{n_j-1})\}.$$

In particular, from $y_j, y_{j+1} \in p_j(a_0, ..., a_{n-1})$ we get

$$u_j \in f(y_j) \subseteq q_j(a_0, ..., a_{n-1}) \supseteq f(y_{j+1}) \ni u_{j+1},$$

proving $u_j\ \alpha\ u_{j+1}$. Thus $u_0\ \alpha^*\ u_{m-1}$. Since $u_0 \in f(a)$ and $u_{m-1} \in f(b)$ were arbitrary, we obtain $f(a) \mathrel{\overline{\overline{\alpha^*}}} f(b)$. □

Recall that, for a given multialgebra $\mathfrak{A}$ and an equivalence relation ρ on A, the set A/ρ can be seen as a multialgebra $\mathfrak{A}/\rho$ with the multioperations:

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle|\ b \in f_\gamma(b_0, ..., b_{n_\gamma-1}), b_i \in \rho\langle a_i\rangle, \forall i \in \{0, ..., n_\gamma - 1\}\},\quad \gamma < o(\tau) \tag{1}$$

(where $\rho\langle x\rangle$ denotes the class of x modulo ρ).

3. Lemma. *If ρ is an equivalence relation on A such that $\mathfrak{A}/\rho$ is a universal algebra then for any $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, ..., a_{n-1} \in A$, we have that $x, y \in p(a_0, ..., a_{n-1})$ implies $x\rho y$.*

Proof. We will prove this statement by induction over the steps of construction of an n-ary polynomial function ($n \in \mathbb{N}$ arbitrary).

If $p = c_a^n$, from $x, y \in c_a^n(a_0, ..., a_{n-1})$ we deduce that $x = y = a$ thus $x\rho y$.

If $p = e_i^n$, with $i \in \{0, ..., n-1\}$, from $x, y \in e_i^n(a_0, ..., a_{n-1})$ we deduce that $x = y = a_i$ thus $x\rho y$.

We suppose that the statement holds for the n-ary polynomial functions $p_0, ..., p_{n_\gamma - 1}$ and we will prove it for the n-ary polynomial function $f_\gamma(p_0, ..., p_{n_\gamma-1})$ (where $\gamma < o(\tau)$). If

$$x, y \in f_\gamma(p_0, ..., p_{n_\gamma-1})(a_0, ..., a_{n-1}) = f_\gamma(p_0(a_0, ..., a_{n-1}), ..., p_{n_\gamma-1}(a_0, ..., a_{n-1}))$$

then there exist $x_i, y_i \in p_i(a_0, ..., a_{n-1})$, $i \in \{0, ..., n_\gamma - 1\}$ such that $x \in f_\gamma(x_0, ..., x_{n_\gamma-1})$ and $y \in f_\gamma(y_0, ..., y_{n_\gamma-1})$. Obviously $x_i \rho y_i$, for all $i \in \{0, ..., n_\gamma - 1\}$ and according to (1) and to the hypothesis of the lemma we have that $\rho\langle x\rangle = \rho\langle y\rangle$, i.e. $x\rho y$. $\square$

Now we can prove the following:

4. Theorem. *The relation α^* is the smallest equivalence relation on A such that $\mathfrak{A}/\alpha^*$ is a universal algebra.*

Proof. To start we show that $\mathfrak{A}/\alpha^*$ is a universal algebra. For this we take any $x, y \in A$ such that $\alpha^*\langle x\rangle, \alpha^*\langle y\rangle \in f_\gamma(\alpha^*\langle a_0\rangle, ..., \alpha^*\langle a_{n_\gamma-1}\rangle)$, with $a_0, ..., a_{n_\gamma-1} \in A$, $\gamma < o(\tau)$; this means that there exist $x_0, ..., x_{n_\gamma-1}, y_0, ..., y_{n_\gamma-1} \in A$ such that $x \in f_\gamma(x_0, ..., x_{n_\gamma-1})$, $y \in$ $\in f_\gamma(y_0, ..., y_{n_\gamma-1})$ and $x_i \ \alpha^* \ a_i \ \alpha^* \ y_i$ for all $i \in \{0, ..., n_\gamma - 1\}$. Applying Lemma 2 to the unary polynomial functions

$$f_\gamma(z, c_{x_1}^n, ..., c_{x_{n_\gamma-1}}^n), f_\gamma(c_{y_0}^n, z, c_{y_2}^n ..., c_{x_{n_\gamma-1}}^n), ..., f_\gamma(c_{y_0}^n, ..., c_{y_{n_\gamma-2}}^n, z)$$

we have the following relations:

$$f_\gamma(x_0, x_1, ..., x_{n_\gamma-1}) \ \overline{\overline{\alpha^*}} \ f_\gamma(y_0, x_1, ..., x_{n_\gamma-1}),$$

$$f_\gamma(y_0, x_1, x_2, ..., x_{n_\gamma-1}) \ \overline{\overline{\alpha^*}} \ f_\gamma(y_0, y_1, x_2, ..., x_{n_\gamma-1}),$$

$$...$$

$$f_\gamma(y_0, ..., y_{n_\gamma-2}, x_{n_\gamma-1}) \ \overline{\overline{\alpha^*}} \ f_\gamma(y_0, ..., y_{n_\gamma-2}, y_{n_\gamma-1}),$$

which leads us (from the definition of $\overline{\overline{\alpha^*}}$) to $x\alpha^* y$, i.e. $\alpha^*\langle x\rangle = \alpha^*\langle y\rangle$. This means that f_γ given in (1) is an operation on A/α^*, for any $\gamma < o(\tau)$, and $\mathfrak{A}/\alpha^*$ is a universal algebra.

Let us show now that any equivalence ρ on A with the property that $\mathfrak{A}/\rho$ is a universal algebra verifies $\alpha^* \subseteq \rho$. Indeed, if $x\alpha y$ then there exist $n \in \mathbb{N}, p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and

$a_0, ..., a_{n_\gamma - 1} \in A$ for which $x, y \in p(a_0, ..., a_{n_\gamma - 1})$; Lemma 3 tells us that $x\rho y$, hence $\alpha \subseteq \rho$, which implies $\alpha^* \subseteq \rho$. $\square$

5. Remarks. a) The equivalence relations ρ on A for which $\mathfrak{A}/\rho$ is a universal algebra are those equivalence relations on A which satisfy the following property: if $a, b \in A$ such that $a\rho b$ then for every $\gamma < o(\tau)$ and $x_0, ..., x_{n_\gamma - 1} \in A$ we have

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma - 1}) \overline{\overline{\rho}} f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma - 1}),$$

for all $i \in \{0, ..., n_\gamma - 1\}$.

Indeed, if $a\rho b$, knowing that $x_j \rho x_j$ for all $j \in \{0, ..., n_\gamma - 1\}$, because any f_γ defined on A/ρ by (1) is an operation it results that

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma - 1}) \overline{\overline{\rho}} f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma - 1}),$$

for all $i \in \{0, ..., n_\gamma - 1\}$. The converse implication can be proved like in the first part of the theorem 4.

b) If we are in the case of the remark a) we can define the operations of the universal algebra $\mathfrak{A}/\rho$ as follows:

$$f_\gamma(\rho\langle a_0 \rangle, ..., \rho\langle a_{n_\gamma - 1} \rangle) = \{\rho\langle b \rangle |\ b \in f_\gamma(a_0, ..., a_{n_\gamma - 1})\}. \tag{2}$$

Moreover, we can write

$$f_\gamma(\rho\langle a_0 \rangle, ..., \rho\langle a_{n_\gamma - 1} \rangle) = \rho\langle b \rangle,\ b \in f_\gamma(a_0, ..., a_{n_\gamma - 1}). \tag{3}$$

6. Examples. 1) If $(H, \circ)$ is a hypergroupoid (i.e. a multialgebra with one binary multioperation) then the equivalence relations ρ satisfying the property that H/ρ with the hyperoperation

$$\rho\langle x \rangle \circ \rho\langle y \rangle = \{\rho\langle z \rangle |\ z \in x' \circ y',\ x' \in \rho\langle x \rangle,\ y' \in \rho\langle y \rangle\}$$

is a groupoid, are the strongly regular equivalence relations on H (see [1], 8, 31), and the operation of H/ρ is

$$\rho\langle x \rangle \circ \rho\langle y \rangle = \rho\langle z \rangle,\ z \in x \circ y.$$

Thus $\overline{\overline{\alpha^*}}$ becomes in this case the smallest strongly regular equivalence on H, i.e. β^* defined in [1], 11.

The same equality of relations can be established by considering the groupoid of polynomial functions of the groupoid of the nonvoid parts of H, $(P^*(H), \circ)$ and writing the definition of α^* in the terms of the polynomial functions of $(P^*(H), \circ)$.

It is useful to observe that if $(H, \circ)$ is a semihypergroup or a hypergroup and ρ is a strongly regular equivalence relation on H (in particular β^*), then $(H/\rho, \circ)$ is a semigroup or a group, respectively (see [1]).

2) Let us consider a hyperring in the general sense $(R, +, \cdot)$. This means that $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup and any $x, y, z \in R$ satisfy $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(y + z) \cdot x \subseteq y \cdot x + z \cdot x$.

Let $(P^*(R), +, \cdot)$ be the universal algebra with two binary operations defined as follows:

$$A + B = \bigcup\{a + b|\ a \in A,\ b \in B\},$$

$$A \cdot B = \bigcup\{a \cdot b|\ a \in A,\ b \in B\}.$$

Obviously, the two operations defined above are associative; moreover,

$$\begin{aligned} A \cdot (B + C) &= \bigcup\{a \cdot (b + c)|\ a \in A, b \in B, c \in C\} \subseteq \\ &\subseteq \bigcup\{a \cdot b + a \cdot c)|\ a \in A, b \in B, c \in C\} \subseteq \\ &\subseteq A \cdot B + A \cdot C \end{aligned}$$

and analogously

$$(B + C) \cdot A \subseteq B \cdot A + C \cdot A.$$

We can construct the universal algebra (with two binary operations) of the polynomial functions of $(P^*(R), +, \cdot)$ for any $n \in \mathbb{N}^*$. The images of the elements of this algebra are the sums of products of nonvoid subsets of R or they are included in the images of some polynomial functions of this form. Thus we can define α on R in the following way:

$$a\alpha b \ \Leftrightarrow\ \exists x_{ij} \in R,\quad i \in \{0, ..., k_j - 1\},\quad j \in \{0, ..., l - 1\}\quad (k_j, l \in \mathbb{N}^*)$$

such that

$$a, b \in \sum_{j=0}^{l-1}(\prod_{i=0}^{k_j-1} x_{ij}). \tag{4}$$

(The left-right implication results from the definition of α and from the remarks from above, and the other implication is trivial.)

If we consider the quotient set R/α^* with the hyperoperations:

$$\alpha^*\langle a\rangle + \alpha^*\langle b\rangle = \{\alpha^*\langle c\rangle|\ c \in a' + b',\ a'\alpha^* a,\ b'\alpha^* b\},$$

$$\alpha^*\langle a\rangle \cdot \alpha^*\langle b\rangle = \{\alpha^*\langle c\rangle|\ c \in a' \cdot b',\ a'\alpha^* a,\ b'\alpha^* b\},$$

which are operations because α^* is a strongly regular equivalence both on $(R, +)$ and $(R, \cdot)$ (see Lemma 2). We have to remark that $(R/\alpha^*, +)$ is a group (abelian, if $(R, +)$ is a commutative hypergroup) and $(R/\alpha^*, \cdot)$ is a semigroup. Let us verify if the distributivity of $\cdot$ with respect to $+$ holds for the universal algebra $(R/\alpha^*, +, \cdot)$. We can rewrite the operations of R/α^*:

$$\alpha^*\langle a\rangle + \alpha^*\langle b\rangle = \alpha^*\langle c\rangle,\ c \in a + b,$$

$$\alpha^*\langle a\rangle \cdot \alpha^*\langle b\rangle = \alpha^*\langle c\rangle,\ c \in a \cdot b,$$

thus

$$\alpha^*\langle a\rangle \cdot (\alpha^*\langle b\rangle + \alpha^*\langle c\rangle) = \alpha^*\langle a\rangle \cdot \alpha^*\langle d\rangle = \alpha^*\langle e\rangle,$$

where $e \in a \cdot d \subseteq a \cdot (b + c) \subseteq a \cdot b + a \cdot c$, therefore

$$\alpha^* \langle e \rangle = \alpha^* \langle x \rangle + \alpha^* \langle y \rangle,$$

where $x \in a \cdot b, y \in a \cdot c$, which leads us to

$$\alpha^* \langle e \rangle = \alpha^* \langle a \rangle \cdot \alpha^* \langle b \rangle + \alpha^* \langle a \rangle \cdot \alpha^* \langle c \rangle.$$

We have proved that

$$\alpha^* \langle a \rangle \cdot (\alpha^* \langle b \rangle + \alpha^* \langle c \rangle) = \alpha^* \langle a \rangle \cdot \alpha^* \langle b \rangle + \alpha^* \langle a \rangle \cdot \alpha^* \langle c \rangle.$$

Analogously we can prove

$$(\alpha^* \langle b \rangle + \alpha^* \langle c \rangle) \cdot \alpha^* \langle a \rangle = \alpha^* \langle b \rangle \cdot \alpha^* \langle a \rangle + \alpha^* \langle c \rangle \cdot \alpha^* \langle a \rangle.$$

We can conclude that, if the hypergroup $(R, +)$ is commutative then $(R/\alpha^*, +, \cdot)$ is a ring.

In the particular case of hyperrings with a zero 0 and a unit 1 the images of the polynomial functions that appear in the definition of the fundamental relation can be completed with zero terms and unit factors such that the definition (4) becomes similar to the definition of $\underline{n}$ from [6].

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