ON THE FUNDAMENTAL ALGEBRA OF A DIRECT PRODUCT OF MULTIALGEBRAS

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Abstract. This paper deals with multialgebras. An important instrument in this paper is the fundamental relation of a multialgebra, which can bring us into the class of the universal algebras. In this paper we will try to establish in what conditions the fundamental algebra of a product of multialgebras is the product of their fundamental algebras.

1. Introduction

Multialgebras (also called hyperstructures) have been studied for more than sixty years and they are used in different areas of mathematics as well as in some applied sciences (see [3]). The class of all the multialgebras of a given type can be seen as a category for which the morphisms are the multialgebra homomorphisms. In [8] we presented some properties of the direct product of multialgebras and we saw that the direct product is the product in this category.

As it results from [13] and [14], an important tool in the hyperstructure theory is the fundamental relation of a multialgebra. From [7] it follows that the factorization of a multialgebra by the fundamental relation furnishes a functor. The question that leads to the results presented in this paper is whether this functor commutes with the products, or, in other words, if the fundamental algebra of a product of multialgebras is the product of their fundamental algebras. As we will see in this paper, the answer is negative, but we can find classes of multialgebras for which this property holds. The semihypergroups from [2, Proposition 346] form a class for which this property holds.

Since the main instruments we are using in our approach are the term functions of the universal algebra of the nonvoid subsets of a multialgebra, we will consider two particular classes of multialgebras for which we know the form of these term functions: the class of hypergroups, and the class of the complete multialgebras (see [7]). As an immediate example of a class which satisfies our property we will obtain the class of the complete hypergroups.

2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence over $\mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let A be a nonempty set and $P^*(A)$ the set of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$

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is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set A of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations $f_\gamma,\ \gamma < o(\tau)$. Of course, the universal algebras are particular cases of multialgebras.

If, for any $\gamma < o(\tau)$ and for any $A_0, \ldots, A_{n_\gamma - 1} \in P^*(A)$, we define

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup \{ f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid\ a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\}\},$$

we obtain a universal algebra on $P^*(A)$ (see [10]). We denote this algebra by $\mathfrak{P}^*(A)$. As in [5], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$.

Let $\mathfrak{A}$ be a multialgebra and ρ be an equivalence relation on its support set A. We obtain, as in [4], a multialgebra on A/ρ by defining the multioperations in the factor multialgebra $\mathfrak{A}/\rho$ as follows: for any $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1}),\ a_i \rho b_i,\ i \in \{0, \ldots, n_\gamma - 1\}\}$$

($\rho\langle x\rangle$ denotes the class of x modulo ρ).

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \ldots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \ldots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \ldots, h(a_{n_\gamma-1})). \tag{1}$$

As in [12] we can see the multialgebra $\mathfrak{A}$ as a relational system $(A, (r_\gamma)_{\gamma<o(\tau)})$ if we consider that, for any $\gamma < o(\tau)$, r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \ldots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow\ a_{n_\gamma} \in f_\gamma(a_0, \ldots, a_{n_\gamma-1}). \tag{2}$$

Thus, the definition of the multialgebra homomorphism follows from the definition of the homomorphism among relational systems.

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it results from [10], the multialgebra isomorphisms can be characterized as being those bijective homomorphisms h for which we have equality in (1).

Remark 1. From the steps of construction of a term (function) it follows that for a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$ then

$$h(p(a_0, \ldots, a_{n-1})) \subseteq p(h(a_0), \ldots, h(a_{n-1})).$$

The definition of the multioperations of $\mathfrak{A}/\rho$ allows us to see the canonical mapping from A to A/ρ as a homomorphism of multialgebras.

The fundamental relation of the multialgebra $\mathfrak{A}$ is the transitive closure $\alpha^* = \alpha^*_{\mathfrak{A}}$ of the relation $\alpha = \alpha_{\mathfrak{A}}$ given on A as follows: for $x, y \in A,\ x\alpha y$ if and only if

$$x, y \in p(a_0, \ldots, a_{n-1}) \text{ for some } n \in \mathbb{N},\ \mathbf{p} \in \mathbf{P}^{(n)}(\tau) \text{ and } a_0, \ldots, a_{n-1} \in A, \tag{3}$$

where $p \in P^{(n)}(\mathfrak{P}^*(A))$ is the term function induced by $\mathbf{p}$ on $\mathfrak{P}^*(A)$. The relation α^* is the smallest equivalence relation on A with the property that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra (see [6] and [7]). The universal algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*$ is called the fundamental algebra of $\mathfrak{A}$. We denote by φ_A the canonical projection of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$ and by $\overline{a}$ the class $\alpha^*\langle a\rangle = \varphi_A(a)$ of an element $a \in A$.

The next theorem is proved in [7] for those homomorphisms for which in (1) we have equality, but this additional request is not used in the proof, so we have:

Theorem 1. *If $\mathfrak{A}$, $\mathfrak{B}$ are multialgebras and $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively, are their fundamental algebras and if $f: A \to B$ is a homomorphism then there exists only one homomorphism of universal algebras $\overline{f}: \overline{A} \to \overline{B}$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ \overline{A} \arrow[r, "\overline{f}"] & \overline{B} \end{tikzcd} \tag{4}$$

(φ_A and φ_B denote the canonical projections).

Corollary 1. *a) If $\mathfrak{A}$ is a multialgebra then $\overline{1_A} = 1_{\overline{A}}$.*

b) If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of the same type τ and if $f: A \to B$, $g: B \to C$ are homomorphisms, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

We can easily construct the category of the multialgebras of the same type τ where the morphisms are considered to be the homomorphisms and the composition of two morphisms is the usual mapping composition. It is known that the universal algebras of the same type τ together with the homomorphisms between them form a category which is, obviously, a full subcategory in the category of the multialgebras introduced above. We will denote by $\mathbf{Malg}(\tau)$ the category of the multialgebras of type τ and by $\mathbf{Alg}(\tau)$ the category of the universal algebras of type τ mentioned before.

Remark 2. From Corollary 1 it results that we can define a functor $F: \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$ as follows: $F(\mathfrak{A}) = \overline{\mathfrak{A}}$, for any multialgebra $\mathfrak{A}$ of type τ, and $F(f) = \overline{f}$ which makes the diagram (4) commutative, for any homomorphism f between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of type τ.

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. Using the model offered by [5] and the definitions of the hyperstructures from [2] and of the generalizations presented in [14], named H_v-structures, we can consider that the n-ary (strong) identity

$$\mathbf{q} = \mathbf{r}$$

is said to be satisfied in multialgebra $\mathfrak{A}$ of type τ if

$$q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1})$$

for all $a_0, \ldots, a_{n-1} \in A$, where q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$. We can also consider that a weak identity (the notation is intended to be as suggestive as possible)

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is said to be satisfied in a multialgebra $\mathfrak{A}$ of type τ if

$$q(a_0, \ldots, a_{n-1}) \cap r(a_0, \ldots, a_{n-1}) \neq \emptyset$$

for all $a_0, \ldots, a_{n-1} \in A$, where q and r have the same signification as before.

Remark 3. Many important particular multialgebras can be defined by using identities.

A hypergroupoid $(H, \circ)$ is a semihypergroup if the identity

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 = \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \tag{5}$$

is satisfied on $(H, \circ)$.

Let H be a nonempty set. A hypergroup $(H, \circ)$ is a semihypergroup which satisfies the reproductive law: $a \circ H = H \circ a = H$, for all $a \in H$. It results that the mappings $/, \backslash \colon H \times H \to P^*(H)$ defined by

$$a/b = \{x \in H \mid a \in x \circ b\},\ b\backslash a = \{x \in H \mid a \in b \circ x\}$$

are two binary multioperations on H. Thus, as we have seen in [7], the hypergroups can be identified with those multialgebras $(H, \circ, /, \backslash)$ for which $H \neq \emptyset$, $\circ$ is associative and the multioperations $/, \backslash$ are obtained from $\circ$ using the above equalities. It results that a semihypergroup $(H, \circ)$ (with $H \neq \emptyset$) is a hypergroup if and only if there exist two binary multioperations $/, \backslash$ on H such that the following weak identities:

$$\mathbf{x}_1 \cap \mathbf{x}_0 \circ (\mathbf{x}_0 \backslash \mathbf{x}_1) \neq \emptyset,\ \mathbf{x}_1 \cap (\mathbf{x}_1/\mathbf{x}_0) \circ \mathbf{x}_0 \neq \emptyset,$$

$$\mathbf{x}_1 \cap \mathbf{x}_0 \backslash (\mathbf{x}_0 \circ \mathbf{x}_1) \neq \emptyset,\ \mathbf{x}_1 \cap (\mathbf{x}_1 \circ \mathbf{x}_0)/\mathbf{x}_0 \neq \emptyset$$

are satisfied on $(H, \circ, /, \backslash)$ (see again [7]).

If we replace above (5) by

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 \cap \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \neq \emptyset, \tag{5'}$$

we obtain the class of the H_v-groups (see [13]).

A mapping $h \colon H \to H'$ between two hypergroups is called hypergroup homomorphism if

$$h(a \circ b) \subseteq h(a) \circ h(b),\ \text{for all } a, b \in H.$$

Clearly, this request makes h a homomorphism between $(H, \circ, /, \backslash)$ and $(H', \circ, /, \backslash)$ since

$$h(a/b) \subseteq h(a)/h(b),\ h(a\backslash b) \subseteq h(a)\backslash h(b),\ \text{for all } a, b \in H.$$

Remark 4. Any (weak or strong) identity satisfied on a multialgebra $\mathfrak{A}$ is satisfied (in a strong manner) in $\overline{\mathfrak{A}}$ (see [7]). So, the fundamental algebra of a hypergroup or of a H_v-group is a group.

Remark 5. Since the hypergroups together with the hypergroup homomorphisms form a category it follows immediately that the mappings from Remark 2 define a functor F from the category **HG** of hypergroups into the category of groups **Grp**.

In [7] we introduced a new class of multialgebras which generalize the notion of complete hypergroup that appears in [2] and that is why we suggested we should name them complete multialgebras. In [7] we proved the following:

Proposition 1. *For a multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type τ, the following conditions are equivalent:*

(i) For all $\gamma < o(\tau)$, for any $a_0, \dots, a_{n_\gamma - 1} \in A$ if $a \in f_\gamma(a_0, \dots, a_{n_\gamma - 1})$ then $\overline{a} = f_\gamma(a_0, \dots, a_{n_\gamma - 1})$, (we identify $\overline{a}$ with $\varphi^{-1}(\overline{a})$ whenever this identification does not create confusion);

(ii) For all $m \in \mathbb{N}$, for any $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, m-1\}\}$ and for any $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in A$, if $q(a_0, \dots, a_{m-1}) \cap r(b_0, \dots, b_{m-1}) \neq \emptyset$ then $q(a_0, \dots, a_{m-1}) = r(b_0, \dots, b_{m-1})$.

Definition 1. A *multialgebra* $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type τ is *complete* if it satisfies one of the two equivalent conditions from **Proposition 1**.

Remark 6. Among all the multialgebras of type τ that can be defined on A, which have the fundamental algebra $\overline{\mathfrak{A}}$, a complete multialgebra is the multialgebra that has the "greatest" multioperations (see [7] and [14]).

Remark 7. We notice that if $\mathfrak{A}$ is a complete multialgebra, then the relation $\alpha_{\mathfrak{A}}$ given by (3) is transitive (so $\alpha_{\mathfrak{A}}^* = \alpha_{\mathfrak{A}}$).

Remark 8. The complete multialgebras of type τ form a subcategory $\mathbf{CMalg}(\tau)$ of $\mathbf{Malg}(\tau)$. So, if we compose F from Remark 2 with the inclusion functor we get a functor (which we will denote by F, too) from $\mathbf{CMalg}(\tau)$ into $\mathbf{Alg}(\tau)$.

3. Direct products of multialgebras

If we have a family of relational systems of the same type $\tau = (n_\gamma + 1)_{\gamma < o(\tau)}$, $(\mathfrak{A}_i = (A_i, (r_\gamma)_{\gamma < o(\tau)}) \mid i \in I)$, in [5] is defined the direct product of this family as being the relational system obtained on the Cartesian product $\prod_{i \in I} A_i$ considering that for $(a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma})_{i \in I} \in \prod_{i \in I} A_i$,

$$((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma})_{i \in I}) \in r_\gamma \Leftrightarrow (a_i^0, \ldots, a_i^{n_\gamma}) \in r_\gamma,\ \forall i \in I.$$

If we consider a family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras of type τ and the relational systems defined by (2), the relational system that results on the Cartesian product $\prod_{i \in I} A_i$ from the above considerations is a multialgebra of type τ with the multioperations:

$$f_\gamma((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I}) = \prod_{i \in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}),$$

for any $\gamma < o(\tau)$. This multialgebra is called the direct product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. We observe that the canonical projections of the product, e_i^I, $i \in I$, are multialgebra homomorphisms.

Proposition 2. [8] *The multialgebra $\prod_{i \in I} \mathfrak{A}_i$ constructed this way, together with the canonical projections, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category $\mathbf{Malg}(\tau)$.*

Lemma 1. [8] *For every $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i \in I}, \ldots, (a_i^{n-1})_{i \in I} \in \prod_{i \in I} A_i$, we have*

$$p((a_i^0)_{i \in I}, \ldots, (a_i^{n-1})_{i \in I}) = \prod_{i \in I} p(a_i^0, \ldots, a_i^{n-1}). \tag{6}$$

Proposition 3. [8] *If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is also satisfied on the multialgebra $\prod_{i \in I} \mathfrak{A}_i$.*

Proposition 4. [8] *If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} = \mathbf{r}$ is also satisfied on the multialgebra $\prod_{i \in I} \mathfrak{A}_i$.*

From Example 3, Proposition 3 and Proposition 4 we have the following:

Corollary 2. *The subcategory* $\mathbf{HG}$ *of* $\mathbf{Malg}((2,2,2))$ *is closed under products.*

Corollary 3. *The direct product of complete multialgebras is a complete multialgebra.*

Indeed, let us consider $m \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, m-1\}\}$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{m-1})_{i\in I}, (b_i^0)_{i\in I}, \ldots, (b_i^{m-1})_{i\in I} \in \prod_{i\in I} A_i$ such that

$$q((a_i^0)_{i\in I}, \ldots, (a_i^{m-1})_{i\in I}) \cap r((b_i^0)_{i\in I}, \ldots, (b_i^{m-1})_{i\in I}) \neq \emptyset.$$

This means, according to Lemma 1, that for any $i \in I$ we have

$$q(a_i^0, \ldots, a_i^{m-1}) \cap r(b_i^0, \ldots, b_i^{m-1}) \neq \emptyset.$$

But all the multialgebras $\mathfrak{A}_i$ are complete, thus

$$q(a_i^0, \ldots, a_i^{m-1}) = r(b_i^0, \ldots, b_i^{m-1})$$

for all $i \in I$, hence

$$q((a_i^0)_{i\in I}, \ldots, (a_i^{m-1})_{i\in I}) = r((b_i^0)_{i\in I}, \ldots, (b_i^{m-1})_{i\in I}).$$

Corollary 4. *The subcategory* $\mathbf{CMalg}(\tau)$ *of* $\mathbf{Malg}(\tau)$ *is closed under products.*

4. On the fundamental algebras of a direct product of multialgebras

Let us consider the universal algebra $\prod_{i\in I} \overline{\mathfrak{A}_i}$ and its canonical projections $p_i : \prod_{i\in I} \overline{A_i} \to \overline{A_i}$ $(i \in I)$. There exists a unique homomorphism φ of universal algebras such that the following diagram is commutative:

$$\begin{tikzcd}
\prod_{i\in I} \overline{\mathfrak{A}_i} \arrow[r, "p_j"] & \overline{\mathfrak{A}_j} \\
\overline{\prod_{i\in I} \mathfrak{A}_i} \arrow[u, "\varphi"] \arrow[ur, "\overline{e_j^I}"'] &
\end{tikzcd}.$$

This homomorphism is given by $\varphi(\overline{(a_i)_{i\in I}}) = (\overline{a_i})_{i\in I}$ for any $(a_i)_{i\in I} \in \prod_{i\in I} A_i$. It is clear that φ is surjective, so the universal algebra $\overline{\prod_{i\in I} \mathfrak{A}_i}$, with the homomorphisms $(\overline{e_i^I} \mid i \in I)$ is the product of the family $(\overline{\mathfrak{A}_i} \mid i \in I)$ if and only if φ is also injective.

But this does not always happen, as it results from the following example.

Example 1. Let us consider the hypergroupoids $(H_1, \circ)$ and $(H_2, \circ)$ on the three elements sets H_1 and H_2 given by the following tables:

H_1	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

H_2	x	y	z
x	x	$y,\ z$	$y,\ z$
y	$y,\ z$	$y,\ z$	$y,\ z$
z	$y,\ z$	$y,\ z$	$y,\ z$

then in $\overline{H_1} \times \overline{H_2}$, $(\bar{b}, \bar{y}) = (\bar{b}, \bar{z})$ but in $\overline{H_1 \times H_2}$ the supposition that $\overline{(b, y)} = \overline{(b, z)}$ leads us to the fact that $y = z$, which is false.

We will use the above notations and we will search for necessary and sufficient conditions expressed with the aid of term functions for φ to be injective. We will deal only with the cases when I is finite or $\alpha_{\mathfrak{A}_i} = \alpha^*_{\mathfrak{A}_i}$ for all $i \in I$ (even if I is not finite).

Lemma 2. *If I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for any $i \in I$, then the homomorphism φ is injective if and only if for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ and for any*

$$(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$$

there exist $m, k_j \in \mathbb{N}$, $\mathbf{q}^j \in \mathbf{P}^{(k_j)}(\tau)$ and $(b_i^0)_{i\in I}^j, \dots, (b_i^{k_j-1})_{i\in I}^j \in \prod_{i\in I} A_i$, $j \in \{0, \dots, m-1\}$ such that

$$(x_i)_{i\in I} \in q^0((b_i^0)_{i\in I}^0, \dots, (b_i^{k_0-1})_{i\in I}^0),\ (y_i)_{i\in I} \in q^{m-1}((b_i^0)_{i\in I}^{m-1}, \dots, (b_i^{k_{m-1}-1})_{i\in I}^{m-1})$$

and

$$q^{j-1}((b_i^0)_{i\in I}^{j-1}, \dots, (b_i^{k_{j-1}-1})_{i\in I}^{j-1}) \cap q^j((b_i^0)_{i\in I}^j, \dots, (b_i^{k_j-1})_{i\in I}^j) \neq \emptyset, \tag{7}$$

for all $j \in \{1, \dots, m-1\}$.

Proof. Let us consider that φ is injective. It is clear that if we take $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ such that

$$(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$$

then we have $(\overline{x_i})_{i\in I} = (\overline{y_i})_{i\in I}$, hence $\overline{(x_i)_{i\in I}} = \overline{(y_i)_{i\in I}}$ and from the definition of the fundamental relation of the multialgebra $\prod_{i\in I} \mathfrak{A}_i$ we get the expected conclusion. Let us remark that this implication does not use the fact that I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for any $i \in I$.

Conversely, if $\alpha_{\mathfrak{A}_i}$ is transitive for any $i \in I$ then $(\overline{x_i})_{i\in I} = (\overline{y_i})_{i\in I}$ if and only if for any $i \in I$ there exists $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ such that $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$, but the condition from the statement is satisfied and, consequently, $\overline{(x_i)_{i\in I}} = \overline{(y_i)_{i\in I}}$. If I is finite then from $(\overline{x_i})_{i\in I} = (\overline{y_i})_{i\in I}$ we will obtain in each $\mathfrak{A}_i$ a chain of l_i term functions (on $\mathfrak{P}^*(A_i)$) such that x_i is in the first term function, y_i is in the last one and every two consecutive members of this chain have a nonempty intersection (we identified here the term functions from our chain as those of their images which interest us). If we consider $l = \max\{l_i \mid i \in I\}$, we can repeat in each chain the last term function as many times as necessary for all our chains to have l members. If we take the Cartesian products of the first members in these chains of term functions, then for the second members and so on, we will obtain l such Cartesian products such that every two consecutive products have a nonempty intersection. For each of them we can apply the condition from the statement of the lemma and we will obtain a chain of term functions in $\prod_{i\in I} \mathfrak{A}_i$ such that $(x_i)_{i\in I}$ is in the first member, $(y_i)_{i\in I}$ is in the last one and any two consecutive members has a nonempty intersection, so $\overline{(x_i)_{i\in I}} = \overline{(y_i)_{i\in I}}$. Thus φ is injective. □

It seems to be a little uncomfortable to work with the condition from the above statement. But we can immediately deduce from this one a sufficient condition which will prove to be useful in the next part of our paper. Of course, we use the same notations and the same hypothesis as before.

Corollary 5. *The condition from the previous lemma is verified if there exist* $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ *and* $b_i^0, \ldots, b_i^{n-1} \in A_i$ $(i \in I)$ *such that*

$$\prod_{i \in I} q_i(a_i^0, \ldots, a_i^{n_i-1}) \subseteq q((b_i^0)_{i \in I}, \ldots, (b_i^{n-1})_{i \in I}). \tag{8}$$

Let us take a subcategory $\mathcal{C}$ of $\mathbf{Malg}(\tau)$ and the functor $F \circ U$ obtained as the composition of the functor F introduced in Remark 2 with the inclusion functor $U : \mathcal{C} \longrightarrow \mathbf{MAlg}(\tau)$. Since we know how U is defined, we can refer to $F \circ U$ as F.

It follows immediately the following statements:

Proposition 5. *Let us consider a subcategory* $\mathcal{C}$ *of* $\mathbf{Malg}(\tau)$ *closed under finite products. Let us also consider that for any finite set* I, *for any family* $(\mathfrak{A}_i \mid i \in I)$ *of multialgebras from* $\mathcal{C}$ *and for any* $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$ $(i \in I)$ *there exist* $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ *and* $b_i^0, \ldots, b_i^{n-1} \in A_i$ $(i \in I)$ *such that (8) holds. Then the functor* $F : \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ *commutes with the finite products.*

Proposition 6. *Let us consider a subcategory* $\mathcal{C}$ *of* $\mathbf{Malg}(\tau)$ *closed under products and let us also consider that* $\alpha_{\mathfrak{A}}$ *is transitive for each* $\mathfrak{A} \in \mathcal{C}$. *Assume that for any set* I, *for any family* $(\mathfrak{A}_i \mid i \in I)$ *of multialgebras from* $\mathcal{C}$ *and for any* $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$ $(i \in I)$ *there exist* $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ *and* $b_i^0, \ldots, b_i^{n-1} \in A_i$ $(i \in I)$ *such that (8) holds. Then the functor* $F : \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ *commutes with the products.*

In the next part of the paper we will study two particular classes of multialgebras for which the relation α defined by (3) is transitive: the class of hypergroups and the class of complete multialgebras. In Subsection 4.1. we will see that the class $\mathcal{C}$ from Proposition 5 can be considered the class of hypergroups. We will also find classes $\mathcal{C}$ of hypergroups as those from Proposition 6 if for an $n \in \mathbb{N}$ we take the class of the hypergroups for which $\beta = \beta_n$ (for notations see [2]). The class of the complete hypergroups is an example in this respect. From Subsection 4.2. we can deduce how we can obtain classes of complete multialgebras which satisfy the conditions from Proposition 6.

4.1. **The case of hypergroups.**

First, let us see what happens for finite products of hypergroups. We remind that the fundamental relation on a hypergroup $(H, \circ, /, \backslash)$ is the transitive closure of the relation $\beta = \bigcup_{n \in \mathbb{N}^*} \beta_n$ where for any $x, y \in H$,

$$x \beta_n y \text{ if and only if there exist } a_0, \ldots, a_{n-1} \in H, \text{ with } x, y \in a_0 \circ \cdots \circ a_{n-1}.$$

The relation β is transitive, so $\beta^* = \beta$ (see [2]). As we can easily see, the term functions q_i which interest us are only those which are involved in the definition of the fundamental relations of the multialgebras $\mathfrak{A}_i$. As it results immediately, in the case of hypergroups, these term functions can be obtained from the canonical projections using only the hyperproduct $\circ$. Any hyperproduct with n factors is a subset of a hyperproduct with $n+1$ factors. This follows from the property of reproducibility of a hypergroup. Indeed, given a hypergroup $(H, \circ, /, \backslash)$, and $a_1, \ldots, a_n \in H$, since $H = H \circ a_1$, there exists an $a_0 \in H$ such that $a_1 \in a_0 \circ a_1$, hence $a_1 \circ \cdots \circ a_n \subseteq a_0 \circ a_1 \circ \cdots \circ a_n$. So $\beta_n \subseteq \beta_{n+1}$, for any $n \in \mathbb{N}^*$. It means that for any two hypergroups $(H_0, \circ)$, $(H_1, \circ)$ we can apply Corollary 5 and it follows that $\overline{H_0 \times H_1}$, together with the homomorphisms $\overline{e_0^2}$, $\overline{e_1^2}$, is the product of the groups $\overline{H_0}$ and $\overline{H_1}$. Thus we have proved the following:

Proposition 7. *The functor* $F\colon \mathbf{HG} \longrightarrow \mathbf{Grp}$ *commutes with the finite products of hypergroups.*

Yet, F does not commute with the arbitrary products of hypergroups, as it follows from the next:

Example 2. Let us consider the hypergroupoid $(\mathbb{Z}, \circ)$, where $\mathbb{Z}$ is the set of the integers and for any $x, y \in \mathbb{Z},\ x \circ y = \{x + y, x + y + 1\}$. It results immediately that $(\mathbb{Z}, \circ)$ is a hypergroup with the fundamental relation $\beta = \mathbb{Z} \times \mathbb{Z}$. It means that the fundamental group of $(\mathbb{Z}, \circ)$ is a one-element group. Now let us consider the product $(\mathbb{Z}^{\mathbb{N}}, \circ)$. The fundamental group of this hypergroup has more than one element. Indeed, $f, g\colon \mathbb{N} \to \mathbb{Z},\ f(n) = 0,\ g(n) = n + 1\ (n \in \mathbb{N})$ are not in the same equivalence class of the fundamental relation of the hypergroup $(\mathbb{Z}^{\mathbb{N}}, \circ)$.

As for arbitrary products (not necessarily finite) of hypergroups we have:

Theorem 2. *For a given set* I *and the hypergroups* $H_i,\ i \in I$, *with the fundamental relations* β^{H_i}, *the group* $\overline{\prod_{i\in I} H_i}$, *together with the homomorphisms* $(\overline{e_i^I} \mid i \in I)$, *is the product of the family of groups* $(\overline{H_i} \mid i \in I)$ *if and only if there exists an* $n \in \mathbb{N}^*$ *such that* $\beta^{H_i} \subseteq \beta_n^{H_i}$, *for all the elements* i *from* I, *except for a finite number of* i*'s.*

Proof. Let us consider $I_n = \{i \in I \mid \beta^{H_i} \not\subseteq \beta_n^{H_i}\}$. It is clear that $I_{n+1} \subseteq I_n$, for any $n \in \mathbb{N}^*$. For a given hypergroup H, the fact that $\beta^H \not\subseteq \beta_n^H$ for an $n \in \mathbb{N}^*$ means that there exist two elements in H which belong to the same hyperproduct with more than n factors but they are not both contained in an hyperproduct with n factors.

Assume there exists $n \in \mathbb{N}^*$ such that I_n is finite. Let us consider an arbitrary family of hyperproducts from the hypergroups $H_i\ (i \in I)$. Each hyperproduct of elements from H_i, with $i \in I \setminus I_n$, is included in a hyperproduct with n factors. Let k be the greatest positive integer which represent the number of factors in the hyperproducts of the given family corresponding to $i \in I_n$. Clearly $k \geq n$, and any hyperproduct of the given family is included in a hyperproduct with k factors, and thus (8) holds.

Conversely, let us consider that for every $n \in \mathbb{N}$ the set I_n is infinite. In order to finish the proof of the theorem, we will construct two families $(a_i)_{i\in I},\ (b_i)_{i\in I}$ from $\prod_{i\in I} H_i$ such that $a_i \beta^{H_i} b_i$, for any $i \in I$, but $((a_i)_{i\in I}, (b_i)_{i\in I}) \notin \beta^{\prod_{i\in I} H_i}$. The construction goes as follows: we choose $i_1 \in I_1$, and we consider $a_{i_1}, b_{i_1} \in H_{i_1}$ such that $a_{i_1} \neq b_{i_1}$ belong to a hyperproduct with more than two factors from H_{i_1}; now, we take $i_2 \in I_2 \setminus \{i_1\}$ and we consider $a_{i_2}, b_{i_2} \in H_{i_2}$ such that a_{i_2}, b_{i_2} are in a hyperproduct with more than three factors from H_{i_2} but they are in no hyperproduct with two factors from H_{i_2}; supposing that we have all the elements a_{i_k}, b_{i_k} for $k \in \mathbb{N}^*,\ k \leq n$, we consider the elements $a_{i_{n+1}}, b_{i_{n+1}} \in H_{i_{n+1}},\ i_{n+1} \in I_n \setminus \{i_1, \ldots, i_n\}$ such that $a_{i_{n+1}}, b_{i_{n+1}}$ are in a hyperproduct with more than $n + 1$ factors from $H_{i_{n+1}}$ but they are in no hyperproduct with n factors from $H_{i_{n+1}}$; for any $i \in I \setminus \{i_n \mid n \in \mathbb{N}^*\}$ we consider $a_i = b_i$. □

Corollary 6. *Let us consider* $n \in \mathbb{N}$. *If* $\mathcal{C}_n$ *is the class of the hypergroups for which* $\beta = \beta_n$ *then* $\mathcal{C}_n$ *is closed under the formation of the direct products and the functor* $F : \mathcal{C}_n \longrightarrow \mathbf{Grp}$ *obtained through factorization with the fundamental relation commutes with the products.*

Since for the complete hypergroups we have $\beta = \beta_2$, we have:

Corollary 7. *The functor F commutes with the products of complete hypergroups.*

4.2. **The case of complete multialgebras.**

It is known that for a complete multialgebra $\mathfrak{A}$ the classes from $\overline{A}$ have the form $\{a\}$ or $f_\gamma(a_0, \dots, a_{n_\gamma - 1})$, with $\gamma < o(\tau)$, $a, a_0, \dots, a_{n_\gamma-1} \in A$ (situations which not exclude each other). We will use this to prove the following:

Theorem 3. *For a family $(\mathfrak{A}_i \mid i \in I)$ of complete multialgebras of the same type τ, the following statements are equivalent:*

i) $\overline{\prod_{i\in I} \mathfrak{A}_i}$ (together with the homomorphisms $\overline{e_i^I}$ $(i \in I)$) is the product of the family of the universal algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$;

ii) For any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$, $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (8) holds with equality;

iii) For any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ either

$$\left| \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) \right| = 1$$

or there exist $\gamma < o(\tau)$, $b_i^0, \dots, b_i^{n_\gamma-1} \in A_i$ $(i \in I)$ such that

$$\prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) = f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}). \tag{9}$$

Proof. $ii) \Leftrightarrow i)$ and $iii) \Rightarrow ii)$ are immediate.

$i) \Rightarrow iii)$ Let us take $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ such that

$$\left| \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) \right| \neq 1$$

and let us consider a family $(x_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$. Then there exists another family $(y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1})$ such that $(x_i)_{i\in I} \neq (y_i)_{i\in I}$. It is clear that $(\overline{x_i})_{i\in I} = (\overline{y_i})_{i\in I}$ and from $i)$ it follows that $\overline{(x_i)_{i\in I}} = \overline{(y_i)_{i\in I}}$. It follows that there exists a $\mathbf{q} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, n-1\}\}$ such that

$$(x_i)_{i\in I}, (y_i)_{i\in I} \in q((c_i^0)_{i\in I}, \dots, (c_i^{n-1})_{i\in I})$$

for some $(c_i^0)_{i\in I}, \dots, (c_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$. Since $\prod_{i\in I} \mathfrak{A}_i$ is a complete multialgebra, there exists a $\gamma < o(\tau)$, and $(b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I} \in \prod_{i\in I} A_i$ such that

$$q((c_i^0)_{i\in I}, \dots, (c_i^{n-1})_{i\in I}) = f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}).$$

Hence $(x_i)_{i\in I} \in f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I})$, which leads us to

$$\prod_{i\in I} q_i(a_i^0, \dots, a_i^{n_i-1}) \cap f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}) \neq \emptyset.$$

But $f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(b_i^0, \dots, b_i^{n_\gamma-1})$ hence

$$q_i(a_i^0, \dots, a_i^{n_i-1}) \cap f_\gamma(b_i^0, \dots, b_i^{n_\gamma-1}) \neq \emptyset.$$

Using the completeness of the multialgebras $\mathfrak{A}_i$ we have

$$q_i(a_i^0, \dots, a_i^{n_i-1}) = f_\gamma(b_i^0, \dots, b_i^{n_\gamma-1})$$

therefore $\prod_{i\in I} q_i(a_i^0,\dots,a_i^{n_i-1}) = \prod_{i\in I} f_\gamma(b_i^0,\dots,b_i^{n_\gamma-1})$ and the equality (9) is satisfied. $\square$

Remark 9. If all $\mathfrak{A}_i$ are universal algebras then *iii*) is trivially satisfied.

Remark 10. For a family of complete multialgebras $(\mathfrak{A}_i \mid i \in I)$ the following conditions are equivalent:

a) there exist $n \in \mathbb{N}$ and $\mathbf{p} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0,..,n-1\}\}$ such that for each $i \in I$ and for any $a_i \in A_i$ we have $a_i \in p(a_i^0,\dots,a_i^{n-1})$ for some $a_i^0,\dots,a_i^{n-1} \in A_i$;

b) there exists a $\gamma < o(\tau)$ such that for each $i \in I$ and for any $a_i \in A_i$ we have $a_i \in f_\gamma(a_i^0,\dots,a_i^{n_\gamma-1})$ for some $a_i^0,\dots,a_i^{n_\gamma-1} \in A_i$.

Corollary 8. *If for a family of complete multialgebras one of the equivalent conditions a) or b) is satisfied, then the condition i) from the previous theorem holds.*

Indeed, let us consider $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0,\dots,a_i^{n_i-1} \in A_i$ $(i \in I)$. For any $i \in I$ we have that $q_i(a_i^0,\dots,a_i^{n_i-1}) \neq \emptyset$, so there exists an $a_i \in q_i(a_i^0,\dots,a_i^{n_i-1})$. But we also have $a_i \in f_\gamma(b_i^0,\dots,b_i^{n_\gamma-1})$ for some $b_i^0,\dots,b_i^{n_\gamma-1} \in A_i$, hence

$$q_i(a_i^0,\dots,a_i^{n_i-1}) \cap f_\gamma(b_i^0,\dots,b_i^{n_\gamma-1}) \neq \emptyset,$$

thus we have that for any $i \in I$,

$$q_i(a_i^0,\dots,a_i^{n_i-1}) = f_\gamma(b_i^0,\dots,b_i^{n_\gamma-1}).$$

It follows that

$$\prod_{i\in I} q_i(a_i^0,\dots,a_i^{n_i-1}) = \prod_{i\in I} f_\gamma(b_i^0,\dots,b_i^{n_\gamma-1}) = f_\gamma((b_i^0)_{i\in I},\dots,(b_i^{n_\gamma-1})_{i\in I}),$$

and (9) is satisfied, thus *i*) holds.

Remark 11. The condition a), respectively b) from above are not necessary for *i*) to be satisfied, and the exception is not covered by the case when all $\mathfrak{A}_i$ are universal algebras.

Example 3. Let us consider the multialgebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$, of the same type (2,3,4) obtained on the sets $A_0 = \{1,2,3\}$, respectively $A_1 = \{1,2,3,4\}$ as follows: $\mathfrak{A}_0 = (A, f_0^0, f_1^0, f_2^0)$, $\mathfrak{A}_1 = (A, f_0^1, f_1^1, f_2^1)$, where $f_j^i\colon A_i^{j+2} \to P^*(A_i)$, $i = 0,1,\ j = 0,1,2$,

$$f_0^0(x,y) = \{1\},\ f_1^0(x,y,z) = \{2,3\},\ f_2^0(x,y,z,t) = \{2,3\},$$

$$f_0^1(x,y) = \{1,2,3\},\ f_1^1(x,y,z) = \{4\},\ f_2^1(x,y,z,t) = \{1,2,3\}.$$

These complete multialgebras satisfy condition *iii*) and, consequently, the condition *i*), but they do not verify condition b).

Remark 12. From Corollary 7 it follows that the complete hypergroups are examples of complete multialgebras for which the required property holds. Of course this also results from Corollary 8.

Remark 13. The complete semihypergroups from [2, Proposition 346] satisfy the condition b) from Remark 10.

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