IDENTITIES AND MULTIALGEBRAS

COSMIN PELEA

Abstract. This paper deals with multialgebras. An important instrument in this paper is the fundamental relation of a multialgebra, which can bring us into the class of the universal algebras. In the first part of the article we will see that the fundamental structure of a multialgebra verifies the identities of the given multialgebra. When trying to obtain multistructures that verify (even in a weak manner) the identities of their fundamental structure we get a new class of multialgebras. In the particular case of the semihypergroups these multialgebras are the complete semihypergroups.

Let $\tau=(n_\gamma)_{\gamma<o(\tau)}$ be a sequence over $\mathbb{N}=\{0,1,\ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma<o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau)=(\mathbf{P}^{(n)}(\tau),(f_\gamma)_{\gamma<o(\tau)})$.

Let $\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)$. According to [4], the n-ary identity $\mathbf{q}=\mathbf{r}$ is said to be satisfied in a class K of universal algebras of type τ if

$$q(a_0,\ldots,a_{n-1})=r(a_0,\ldots,a_{n-1}),$$

for all $a_0,\ldots,a_{n-1}\in A$ and for all $\mathfrak{A}\in K$, (q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{A}$.)

Let A be a nonvoid set and $P^*(A)$ the set of the nonempty subsets of A. Let $\mathfrak{A}=(A,(f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where $f_\gamma:A^{n_\gamma}\to P^*(A)$ is the multioperation of arity $n_\gamma\in\mathbb{N}$ that corresponds to the symbol $\mathbf{f}_\gamma$, for any $\gamma<o(\tau)$. The multialgebra $\mathfrak{A}$ induces a universal algebra $(P^*(A),(f_\gamma)_{\gamma<o(\tau)})$ with the operations:

$$f_\gamma(A_0,\ldots,A_{n_\gamma-1})=\bigcup\{f_\gamma(a_0,\ldots,a_{n_\gamma-1})\mid\ a_i\in A_i,\ i\in\{0,\ldots,n_\gamma-1\}\},$$

for any γ<o(τ) and $A_0,\ldots,A_{n_\gamma-1}\in P^*(A)$ (see [7]). We denote this algebra by $\mathfrak{P}^*(A)$.

In [4], Grätzer presents the algebra of the term functions of a universal algebra $\mathfrak{B}=(B,(f_\gamma)_{\gamma<o(\tau)})$. For any $n\in\mathbb{N}$, we can construct the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of n-ary term functions on $\mathfrak{P}^*(A)$.

Consider the set $P_A^{(n)}(\mathfrak{P}^*(A))$ ($n\in\mathbb{N}$) of those and only those functions from $(P^*(A))^n$ into $P^*(A)$ which can be obtained from (i), (ii) and (iii) from bellow in a finite number of steps:

(i) for every $a\in A$, the function

$$c_a^n:(P^*(A))^n\to(P^*(A)),\ c_a^n(X_0,\ldots,X_{n-1})=a$$

($X_0,\ldots,X_{n-1}\in P^*(A)$) is an element of $P_A^{(n)}(\mathfrak{P}^*(A))$;

(ii) for any $i=0,\ldots,n-1$, the function

$$e_i^n:(P^*(A))^n\to P^*(A),\ e_i^n(X_0,\ldots,X_{n-1})=X_i$$

($X_0,\ldots,X_{n-1}\in P^*(A)$) is an element of $P_A^{(n)}(\mathfrak{P}^*(A))$;

(iii) if $p_0,\dots,p_{n_\gamma-1}$ are elements of $P_A^{(n)}(\mathfrak{P}^*(A))$ and $\gamma<o(\tau)$ then the function $p=f_\gamma(p_0,\dots,p_{n_\gamma-1}):(P^*(A))^n\to P^*(A)$ defined by

$$p(X_0,\dots,X_{n-1})=f_\gamma(p_0(X_0,\dots,X_{n-1}),\dots,p_{n_\gamma-1}(X_0,\dots,X_{n-1}))$$

is also an element of $P_A^{(n)}(\mathfrak{P}^*(A))$.

In [5], one defines the fundamental relation of the multialgebra $\mathfrak{A}$ as the transitive closure α^* of the relation α given on A as follows: for $x,y\in A$, $x\alpha y$ if and only if

$$x,y\in p(a_0,\dots,a_n) \text{ for some } n\in\mathbb{N},\ p\in P_A^{(n)}(\mathfrak{P}^*(A)) \text{ and } a_0,\dots,a_n\in A. \tag{1}$$

It is easy to observe that, in this definition, we can consider $P^{(n)}(\mathfrak{P}^*(A))$ instead of $P_A^{(n)}(\mathfrak{P}^*(A))$ and the relation α remains the same. The relation α^* is the smallest equivalence relation on A with the property that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra. For the sake of brevity let us denote the algebra $\mathfrak{A}/\alpha^*$ by $\overline{\mathfrak{A}}$ (and A/α^* by $\overline{A}$) and let us call it fundamental algebra of the multialgebra $\mathfrak{A}$.

We can remember that the definition of the multioperations in the factor multialgebra $\mathfrak{A}/\rho$ (ρ is an equivalence on A) is the same as in [3]:

$$\begin{aligned} f_\gamma(\rho\langle a_0\rangle,\dots,\rho\langle a_{n_\gamma-1}\rangle)=\{\rho\langle b\rangle\mid b\in f_\gamma(b_0,\dots,b_{n_\gamma-1}),\ b_i\in\rho\langle a_i\rangle,\\ i\in\{0,\dots,n_\gamma-1\}\},\quad \gamma<o(\tau)\end{aligned} \tag{2}$$

(where $\rho\langle x\rangle$ denotes the class of x modulo ρ). The definition of the multioperations of $\mathfrak{A}/\rho$ allows us to see the canonical map from A to A/ρ as an ideal homomorphism of multialgebras whenever ρ is an ideal equivalence on A (see [7]). We will drop the adjective 'ideal' because all our homomorphisms will be ideal.

The factorization with the fundamental relation of a multialgebra have a functorial character as we can deduce from the following:

Theorem 1. *If $\mathfrak{A}$, $\mathfrak{B}$ are multialgebras and $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively are their fundamental algebras and if $f:A\to B$ is a homomorphism then there exists only one homomorphism $\overline{f}:\overline{A}\to\overline{B}$ so that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ \overline{A} \arrow[r, "\overline{f}"] & \overline{B} \end{tikzcd}$$

where φ_A and φ_B are the canonical projections.

Proof. If we consider that such an $\overline{f}$ exists, it is defined by

$$\overline{f}(\alpha^*\langle a\rangle)=\beta^*\langle f(a)\rangle, \tag{3}$$

for any $a\in A$, where α^* and β^* denote the fundamental relations of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Thus the uniqueness of $\overline{f}$ is proved.

Let us show that $\overline{f}$ exists. We consider $\overline{f}:\overline{A}\to\overline{B}$ given by (2). The application $\overline{f}$ is well defined. Indeed, let $x,y\in A$ be such that $x\alpha^*y$, i.e. there exist $m\in\mathbb{N}$, and $x=x_0,x_1,\dots,x_m=y\in A$ with $x_i\alpha x_{i+1}$ for all $i\in\{1,\dots,m-1\}$, thus for all $i\in\{1,\dots,m-1\}$, there exist $k_i\in\mathbb{N}$, $p_i\in P^{(k_i)}(\mathfrak{P}^*(A))$ and $a_0^i,\dots,a_{k_i-1}^i\in A$ such that $x_i,x_{i+1}\in p_i(a_0^i,\dots,a_{k_i-1}^i)$. Then we have

$$f(x_i),f(x_{i+1})\in p_i(f(a_0^i),\dots,f(a_{k_i-1}^i))$$

(see [1]) and, consequently, $f(x_i)\beta f(x_{i+1})$, for all $i \in \{1, \dots, m-1\}$. It follows that $f(x) = f(x_0)\beta^* f(x_m) = f(y)$.

Now we will verify that $\overline{f}$ is an homomorphism. If $\gamma < o(\tau)$ and $a_0, \dots, a_{n_\gamma - 1} \in A$ we have:

$$\overline{f}(f_\gamma(\alpha^*\langle a_0\rangle, \dots, \alpha^*\langle a_{n_\gamma-1}\rangle)) = \overline{f}(\alpha^*\langle a\rangle) = \beta^*\langle f(a)\rangle,$$

where $a \in f_\gamma(a_0, \dots, a_{n_\gamma-1})$.

Since f is an homomorphism, $f(a) \in f_\gamma(f(a_0), \dots, f(a_{n_\gamma-1}))$ and

$$\beta^*\langle f(a)\rangle = f_\gamma(\beta^*\langle f(a_0)\rangle, \dots, \beta^*\langle f(a_{n_\gamma-1})\rangle) = f_\gamma(\overline{f}(\alpha^*\langle a_0\rangle), \dots, \overline{f}(\alpha^*\langle a_{n_\gamma-1}\rangle))$$

and the theorem is proved. □

Corollary 1. *If $\mathfrak{A}$ is a multialgebra then $\overline{1_A} = 1_{\overline{A}}$.*

Corollary 2. *If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of the same type τ and if $f : A \to B$, $g : B \to C$ are homomorphisms, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.*

For a given multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ we will use the following notations: $\overline{\mathfrak{A}}$ for the fundamental algebra, α^* (or $\alpha^*_{\mathfrak{A}}$ when it is necessary) for the fundamental relation on $\mathfrak{A}$ (* means that we take the transitive closure of the relation α defined by (1)) and φ_A for the canonical projection of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$. We will also denote by $\overline{a}$ the class $\alpha^*\langle a\rangle = \varphi_A(a)$, for any $a \in A$.

Looking at the definitions of the hyperstructures from [2] and also at the generalizations presented in [8], named H_v-structures, we can consider in a similar way that the n-ary identity

$$\mathbf{q} = \mathbf{r}$$

is said to be satisfied in a class K of multialgebras of type τ if

$$q(a_0, \dots, a_{n-1}) = r(a_0, \dots, a_{n-1}),$$

for all $a_0, \dots, a_{n-1} \in A$ and for all $\mathfrak{A} \in K$ (q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$). We can also consider that a weak identity (the notation is intended to be as suggestive as possible)

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is said to be satisfied in a class K of multialgebras of type τ if

$$q(a_0, \dots, a_{n-1}) \cap r(a_0, \dots, a_{n-1}) \neq \emptyset,$$

for all $a_0, \dots, a_{n-1} \in A$ and for all $\mathfrak{A} \in K$ (q and r have the same signification as before).

From [1] it results that

$$\varphi_A(p(a_0, \dots, a_{n-1})) = p(\overline{a_0}, \dots, \overline{a_{n-1}}), \tag{4}$$

for any $n \in \mathbb{N}$, any $\mathbf{p} \in P^{(n)}(\tau)$ and any $a_0, \dots, a_{n-1} \in A$ (p denotes the term function induced by $\mathbf{p}$ on $\mathfrak{P}^*(A)$).

Considering $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ such that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ on $\mathfrak{A}$, i.e.

$$q(a_0, \dots, a_n) \cap r(a_0, \dots, a_n) \neq \emptyset, \ \forall a_0, \dots, a_n \in A,$$

it results that there exists an element

$$a \in q(a_0, \dots, a_n) \cap r(a_0, \dots, a_n)$$

and, according to (4), we have

$$\overline{a} = q(\overline{a_0}, \dots, \overline{a_{n-1}}) = r(\overline{a_0}, \dots, \overline{a_{n-1}}).$$

Thus we have proved the following:

Proposition 1. *If $\mathfrak{A}$ is a multialgebra and $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ such that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{A}$ then $\mathbf{q} = \mathbf{r}$ is satisfied on $\overline{\mathfrak{A}}$.*

Corollary 3. *Let K be a class of multialgebras, and let $\overline{K}$ be the class of the fundamental algebras of the multialgebras from K. If $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied in K then $\mathbf{q} = \mathbf{r}$ is satisfied in $\overline{K}$.*

Remark 1. Considering a family $\{\mathfrak{A}_i\}_{i \in I}$ of multialgebras or type τ, we can organize the Cartesian product $\prod_{i \in I} A_i$ as a multialgebra of type τ with the multioperations defined as it follows:

$$f_\gamma((a_i^0)_{i \in I}, \dots, (a_i^{n_\gamma - 1})_{i \in I}) = \prod_{i \in I} f_\gamma(a_i^0, \dots, a_i^{n_\gamma - 1}), \tag{5}$$

for any $\gamma < o(\tau)$. We observe that the canonical projections of the product are homomorphisms of multialgebras. Let us call variety of multialgebras a class of multialgebras closed under the formation of submultialgebras, homomorphic images and direct products. It is obvious that if K is a variety of multialgebras then K includes the class $\overline{K}$ because each fundamental algebra is a homomorphic image of the multialgebra that determines it, thus the class $\overline{K}$ is a variety of universal algebras.

Remark 2. If K is a variety of multialgebras, Σ is a set of weak or/and strong identities and K_Σ is the subclass of K with elements multialgebras that verify the identities from Σ, then K_Σ is also a variety of multialgebras. This way, many of the hyperstructures can be seen as varieties. For instance, we can see the class of the canonical hypergroups as a subclass in the class of the hypergroups characterized by some identities and it will result that the canonical hypergroups form a variety.

We start by identifying a hypergroup $(H, \circ)$ with a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations, with $H \neq \emptyset$, with $\circ$ associative and

$$a/b = \{x \in H \mid a \in x \circ b\},\ b\backslash a = \{x \in H \mid a \in b \circ x\},\ \forall a, b \in H.$$

The subalgebras of the multialgebra $(H, \circ, /, \backslash)$ are the closed subhypergroups of the hypergroup H. The (ideal) homomorphisms in our case are the very good homomorphisms. It results that the homomorphic image of such a multialgebra has the same properties, hence it is a hypergroup. The direct product of hypergroups is a hypergroup. Moreover, the multioperations $/$ and $\backslash$ defined on the Cartesian product by (5) are the same as those obtained from $\circ$ by using the above equalities. So, the hypergroups form a variety.

Let us notice that a semihypergroup $(H, \circ)$ (with $H \neq \emptyset$) is a hypergroup if and only if there exist two binary multioperations $/, \backslash$ on H such that

$$b \in a \circ (a\backslash b),\ b \in (b/a) \circ a,\quad b \in a\backslash(a \circ b),\ b \in (b \circ a)/a,\ \forall a, b \in H.$$

The existence of $/$ and $\backslash$ need not mean that $x \in a/b$ ($x \in b\backslash a$) if and only if $a \in x \circ b$ ($a \in b \circ a$). However, the above considerations allows us to see the class of hypergroups as a subclass (strictly included) in the class of the multialgebras of type $(2, 2, 2)$ which verify the associativity and the above identities. An immediate consequence is the fact that the fundamental algebra of a hypergroup is a group.

The canonical hypergroups form a variety because we can see them as multialgebras $(H, \circ, /, \backslash, e, ')$ with $(H, \circ, /, \backslash)$ hypergroup, e nullary (multi)operation and $'$ unary (multi)operation, which verify the following identities:
(a) $a \circ b = b \circ a,\ \forall a, b \in H$;
(b) $e \circ a = a (= a \circ e), \forall a \in A$;
(c) $a/b = (b/a)'\ (a\backslash b = (b\backslash a)'),\ \forall a, b \in H$.

In general, Proposition 1 does not work with equivalence. A good example is when the fundamental algebra has only one element, but this is not the only situation. Yet, starting from a universal algebra with more than one element, we can construct multialgebras with a given fundamental algebra which verify the identities of the fundamental algebra, some of them in a weak manner, some of them in a strong manner.

Proposition 2. *Let $\mathfrak{A}$ be a multialgebra and let $\overline{\mathfrak{A}}$ be its fundamental algebra. If $|\overline{A}| > 1$, $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ and $\mathbf{q} = \mathbf{r}$ is satisfied on $\overline{\mathfrak{A}}$ then there exists a multistructure of multialgebra of type τ, $\mathfrak{A}'$, on A, with the multioperations $(f'_\gamma)_{\gamma < o(\tau)}$ such that $\overline{\mathfrak{A}'} = \overline{\mathfrak{A}}$ and $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{A}'$.*

Proof. Let us define

$$f'_\gamma(a_0, \dots, a_{n_\gamma - 1}) = \{a \in A \mid \overline{a} = f_\gamma(\overline{a_0}, \dots, \overline{a_{n_\gamma - 1}})\}, \tag{6}$$

for any $\gamma < o(\tau)$ and for any $a_0, \dots, a_{n_\gamma - 1} \in A$, and let us take $\mathfrak{A}' = (A, (f'_\gamma)_{\gamma < o(\tau)})$.

We start by proving that for any $n \in \mathbb{N}$, $\mathbf{p} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, n-1\}\}$ and for any $a_0, \dots, a_{n-1} \in A$ we have

$$p'(a_0, \dots, a_{n-1}) = \{a \in A \mid \overline{a} = p(\overline{a_0}, \dots, \overline{a_{n-1}})\}, \tag{7}$$

where p' denotes the term function induced on $\mathfrak{A}'$ by $\mathbf{p}$.

Let us observe that for any $\gamma < o(\tau)$, $\forall a_0, \dots, a_{n_\gamma - 1} \in A$

$$f_\gamma(a_0, \dots, a_{n_\gamma - 1}) \subseteq f'_\gamma(a_0, \dots, a_{n_\gamma - 1}).$$

This allows us to verify that for each $n \in \mathbb{N}$ and $\mathbf{p} \in P^{(n)}(\tau)$ we have:

$$p(a_0, \dots, a_{n-1}) \subseteq p'(a_0, \dots, a_{n-1}),\ \forall a_0, \dots, a_{n-1} \in A. \tag{8}$$

From the definition (6) of the multioperations f'_γ, $\gamma < o(\tau)$, it follows that $p'(a_0, \dots, a_{n-1})$ is a class from $\overline{A}$ whenever $\mathbf{p} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, n-1\}\}$ (we identified $\overline{a}$ with $\varphi_A^{-1}(\overline{a})$). Using (8) one obtains (7).

Let us consider the fundamental relation $\alpha^*_{\mathfrak{A}'}$ on the multialgebra $\mathfrak{A}'$. We can write $x \alpha_{\mathfrak{A}'} y$ iff there exist $m \in \mathbb{N}$, $\mathbf{p} \in P^{(m)}(\tau)$, $a_0, \dots, a_{m-1} \in A$ so that $x, y \in p'(a_0, \dots, a_{m-1})$. If $\mathbf{p} = \mathbf{x}_i$ for some $i \in \{0, \dots, m-1\}$ then $x = y$ hence $x \alpha_{\mathfrak{A}} y$ and if $\mathbf{p} \neq \mathbf{x}_i$, $\forall i \in \{0, \dots, m-1\}$ then $x, y \in p'(a_0, \dots, a_{m-1}) \in A$ implies $x \alpha^*_{\mathfrak{A}} y$, because x and y are in the same class from $\overline{A}$. It is clear now that $\alpha^*_{\mathfrak{A}'} \subseteq \alpha^*_{\mathfrak{A}}$. The following implications justify the inverse inclusion: $x \alpha_{\mathfrak{A}} y$ implies the existence of $m \in \mathbb{N}$, $\mathbf{p} \in P^{(m)}(\tau)$ and $a_0, \dots, a_{m-1} \in A$ with $x, y \in p(a_0, \dots, a_{m-1})$; according to (8) we have $x, y \in p'(a_0, \dots, a_{m-1})$ and so $x \alpha_{\mathfrak{A}'} y$. We get that $\alpha^*_{\mathfrak{A}'} = \alpha^*_{\mathfrak{A}}$. We also have $\forall \gamma < o(\tau)$, $\forall a_0, \dots, a_{n_\gamma - 1} \in A$, $f'_\gamma(\overline{a_0}, \dots, \overline{a_{n_\gamma - 1}}) = \overline{a}$ with $a \in f'_\gamma(a_0, \dots, a_{n_\gamma - 1})$, but $f_\gamma(a_0, \dots, a_{n_\gamma - 1}) \subseteq f'_\gamma(a_0, \dots, a_{n_\gamma - 1})$ thus $f'_\gamma(\overline{a_0}, \dots, \overline{a_{n_\gamma - 1}}) = f_\gamma(\overline{a_0}, \dots, \overline{a_{n_\gamma - 1}})$. It is now proved that $\overline{\mathfrak{A}'} = \overline{\mathfrak{A}}$.

Let us consider now $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ with

$$q(\overline{a_0}, \dots, \overline{a_{n-1}}) = r(\overline{a_0}, \dots, \overline{a_{n-1}}),$$

for all $a_0, \ldots, a_{n-1} \in A$. From $|\overline{A}| \neq 1$ it results that $\overline{x}, \overline{y} \in \overline{A}$ exist so that $\overline{x} \neq \overline{y}$ and so $\mathbf{q} = \mathbf{x}_i, \mathbf{r} = \mathbf{x}_j$ implies $i = j$ and in this case the property holds in a trivial manner. If $\mathbf{q} = \mathbf{x}_i$ and $\mathbf{r} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ then $\overline{a_i} = r(\overline{a_0}, \ldots, \overline{a_{n-1}})$ leads us, according to (7), to $a_i \in r'(a_0, \ldots, a_{n-1})$ and the property in the statement holds. If both $\mathbf{q}$ and $\mathbf{r}$ are in $P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ then, using (7), from $q(\overline{a_0}, \ldots, \overline{a_{n-1}}) = r(\overline{a_0}, \ldots, \overline{a_{n-1}})$ we get that $q'(a_0, \ldots, a_{n-1}) = r'(a_0, \ldots, a_{n-1})$ and the proof is accomplished. □

Remark 3. Using the notations above, if $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ and $\mathbf{q} = \mathbf{r}$ on $\overline{\mathfrak{A}}$ then $\mathbf{q} = \mathbf{r}$ on $\mathfrak{A}'$.

Remark 4. The multialgebra $\mathfrak{A}'$ could be defined even if $|A| = 1$, but there will appear some problems about the identities satisfied on $\overline{\mathfrak{A}'} = \overline{\mathfrak{A}}$ which are also satisfied on $\mathfrak{A}'$ (on $\overline{\mathfrak{A}}$, the identities $\mathbf{x}_i = \mathbf{x}_j$ with $i \neq j$ are satisfied, but they are not necessarily satisfied on $\mathfrak{A}'$). Yet, if $\mathbf{q}$ or $\mathbf{r}$ are in $P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ and $\mathbf{q} = \mathbf{r}$ on $\overline{\mathfrak{A}}$ then $\mathbf{q} = \mathbf{r}$ on $\mathfrak{A}'$.

Remark 5. If we generalize $\leq$, presented for hyperproducts in [8], to multioperations then, for any multialgebra $\mathfrak{A}'' = (A, (f''_\gamma)_{\gamma < o(\tau)})$ with $\overline{\mathfrak{A}''} = \overline{\mathfrak{A}}$, we have $f''_\gamma \leq f'_\gamma$, $\forall \gamma < o(\tau)$.

Remark 6. The classes from $\overline{A}$ are of the form $\{a\}$ or $f'_\gamma(a_0, \ldots, a_{n_\gamma - 1})$.

Remark 7. The fundamental relation of the multialgebra $\mathfrak{A}'$ has the following property: $\forall \gamma < o(\tau)$, $\forall a_0, \ldots, a_{n_\gamma - 1} \in A$, $a \in f'_\gamma(\overline{a_0}, \ldots, \overline{a_{n_\gamma - 1}}) \Rightarrow \overline{a} = f'_\gamma(\overline{a_0}, \ldots, \overline{a_{n_\gamma - 1}})$, thus $\alpha^*_{\mathfrak{A}'}$ verifies a generalization of the property which defines, in [2], the notion of congruence (for semihypergroups).

The properties of the multialgebra $\mathfrak{A}'$ suggests the construction of a new class of multialgebras.

Proposition 3. *The following conditions are equivalent for a multialgebra* $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ *of type* τ *:*

(i) for all $\gamma < o(\tau)$*, for all* $a_0, \ldots, a_{n_\gamma - 1} \in A$,

$$a \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) \Rightarrow \overline{a} = f_\gamma(a_0, \ldots, a_{n_\gamma - 1}),$$

(we identify $\overline{a}$ *with* $\varphi^{-1}(\overline{a})$ *whenever this identification does not create confusion).*

(ii) for all $m \in \mathbb{N}$*, for all* $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, m-1\}\}$*, for all* $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in A$,

$$q(a_0, \ldots, a_{m-1}) \cap r(b_0, \ldots, b_{m-1}) \neq \emptyset \Rightarrow q(a_0, \ldots, a_{m-1}) = r(b_0, \ldots, b_{m-1}).$$

Proof. *(i)* $\Rightarrow$ *(ii)*. If (i) holds for $\mathfrak{A}$ then $\forall m \in \mathbb{N}$, $\forall \mathbf{p} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, m-1\}\}$, $\forall a_0, \ldots, a_{m-1} \in A$, we have

$$a \in p(a_0, \ldots, a_{m-1}) \Leftrightarrow \overline{a} \in p(a_0, \ldots, a_{m-1}),$$

which justify (ii).

(ii) $\Rightarrow$ *(i)*. Let us consider $\gamma < o(\tau)$, $a_0, \ldots, a_{n_\gamma - 1} \in A$, $a \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1})$ and $b \in A$ with $b \in \overline{a}$ (i.e. $a\alpha^* b$). It follows that $n \in \mathbb{N}$, $x_0, \ldots, x_n \in A$ exist so that $a = x_0 \alpha x_1 \alpha \ldots \alpha x_{n-1} \alpha x_n = b$ hence for any $i \in \{0, \ldots, n-1\}$, there exist $m_i \in \mathbb{N}$, $\mathbf{p}_i \in P^{(m_i)}(\tau)$, $a^i_0, \ldots, a^i_{m_i - 1} \in A$ with $x_i, x_{i+1} \in p_i(a^i_0, \ldots, a^i_{m_i - 1})$. We can consider that every two consequent elements from $x_0, \ldots, x_n$ are distinguished, thus no $\mathbf{p}_i$ is equal to an $\mathbf{x}^i_j$, $(j < m_i)$, $\forall i \in \{0, \ldots, n-1\}$. Hence

$\forall i \in \{0, \ldots, n-1\}$, $p_i(a_0^i, \ldots, a_{m_i-1}^i) \cap p_{i+1}(a_0^{i+1}, \ldots, a_{m_i-1}^{i+1}) \neq \neq \emptyset$ (because this intersection contains x_i) and so $\forall i \in \{0, \ldots, n-1\}$, $p_i(a_0^i, \ldots, a_{m_i-1}^i) = p_{i+1}(a_0^{i+1}, \ldots, a_{m_i-1}^{i+1})$ which leads us to $b \in p_0(a_0^0, \ldots, a_{m_i-1}^0)$. But

$$a \in p_0(a_0^0, \ldots, a_{m_i-1}^0) \cap f_\gamma(a_0, \ldots, a_{n_\gamma-1})$$

thus $p_0(a_0^0, \ldots, a_{m_i-1}^0) = f_\gamma(a_0, \ldots, a_{n_\gamma-1})$ hence $b \in f_\gamma(a_0, \ldots, a_{n_\gamma-1})$ and the proof is finished. □

Remark 8. A multialgebra $\mathfrak{A}$ which verifies one of the equivalent conditions (i) and (ii) from above is a generalization of the notion of complete semihypergroup from [2] (fact that suggests a name like complete multialgebra).

Remark 9. For a multialgebra $\mathfrak{A}$ which verifies (i) the classes from $\overline{A}$ have the form $\{a\}$ or $f_\gamma(a_0, \ldots, a_{n_\gamma-1})$, with $\gamma < o(\tau)$.

Remark 10. The multialgebra $\mathfrak{A}'$ from the proof of Proposition 2 verifies (i).

Remark 11. The equality $\alpha_{\mathfrak{A}} = \alpha^*_{\mathfrak{A}}$ holds, for any multialgebra $\mathfrak{A}$ which verifies (i).

Proposition 4. *Let $\mathfrak{A}$ be a multialgebra of type τ. The multialgebra $\mathfrak{A}$ verifies (i) if and only if A is a disjoint union of nonvoid sets A_b, $b \in B$, where B is the supporting set of a universal algebra $\mathfrak{B}$ of type τ and for each $\gamma < o(\tau)$, for any $a_0, \ldots, a_{n_\gamma-1} \in A$, with $a_i \in A_{b_i}$ $(i \in \{0, \ldots, n_\gamma - 1\})$, we have $f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = A_{f_\gamma(b_0, \ldots, b_{n_\gamma-1})}$.*

Proof. Let us take $\mathfrak{B} = \overline{\mathfrak{A}}$ and $A_b = \varphi_A^{-1}(b)$, for any $b \in B$. For any $\gamma < o(\tau)$, $a_0, \ldots, a_{n_\gamma-1} \in A$, there exist $b_0, \ldots, b_{n_\gamma-1} \in B$ uniquely determined, with $a_i \in A_{b_i}$, for all $i \in \{0, \ldots, n_\gamma - 1\}$ and we have that $f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \varphi_A^{-1}(\overline{a})$, for each $a \in f_\gamma(a_0, \ldots, a_{n_\gamma-1})$ hence

$$\begin{aligned}\overline{a} = \varphi_A(a) = \varphi_A(f_\gamma(a_0, \ldots, a_{n_\gamma-1})) &= f_\gamma(\varphi_A(a_0), \ldots, \varphi_A(a_{n_\gamma-1})) \\ &= f_\gamma(b_0, \ldots, b_{n_\gamma-1}),\end{aligned}$$

and so $f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \varphi_A^{-1}(f_\gamma(b_0, \ldots, b_{n_\gamma-1})) = A_{f_\gamma(b_0, \ldots, b_{n_\gamma-1})}$.

Conversely, let us notice that for any $n \in \mathbb{N}$, $\mathbf{p} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$, $a_0, \ldots, a_{n-1} \in A$ if $b_0, \ldots, b_{n-1} \in B$ so that $a_i \in A_{b_i}$, $\forall i \in \{0, \ldots, n-1\}$ we have that $p(a_0, \ldots, a_{n-1}) = A_{p(b_0, \ldots, b_{n-1})}$. Using (ii) we obtain the wanted result. □

Remark 12. This proposition gives us a method to obtain multialgebras that verify (i). The fundamental algebra of such a multialgebra contains $\mathfrak{B}$ as a subalgebra.

Remark 13. We can construct multialgebras $\mathfrak{A}$ with the fundamental algebra $\mathfrak{B}$ considering that, in the family $\{A_b\}_{b \in B}$, $|A_b| = 1$ holds for each $b \in B$ which can not be expressed as $b = f_\gamma(b_0, \ldots, b_{n_\gamma-1})$ with $\gamma < o(\tau)$ and $b_0, \ldots, b_{n_\gamma-1} \in B$.

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"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu nr. 1, RO-3400 Cluj-Napoca, Romania

E-mail address: `cpelea@math.ubbcluj.ro`