A CHARACTERIZATION THEOREM FOR COMPLETE MULTIALGEBRAS

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Abstract. In this paper we will give a characterization for the complete multialgebras, which involves universal algebras and their congruences.

The starting point of this paper is in [8] where a class of multialgebras has been introduced. In [8] is proved that the fundamental algebra of a multialgebra $\mathfrak{A}$ verifies the identities which are satisfied (even in a weak manner) on $\mathfrak{A}$. The class of multialgebras we study in this paper appeared while trying to obtain multialgebras which satisfy (at least in a weak manner) the identities verified on their fundamental algebras (see again [8]). Since in the particular case of the (semi)hypergroups these multialgebras are the complete (semi)hypergroups from [3] and [4], these multialgebras were called complete. We will prove that such a multialgebra can be obtained from a universal algebra and an appropriate congruence on it.

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence with $n_\gamma \in \mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let A be a set and $P^*(A)$ the set of the nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set A of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations $f_\gamma,\ \gamma < o(\tau)$. Of course, any universal algebra is a multialgebra (we can identify an one element set with its element).

Let us define for any $\gamma < o(\tau)$ and for any $A_0, \ldots, A_{n_\gamma-1} \in P^*(A)$

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid\ a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\}\}.$$

We obtain a universal algebra on $P^*(A)$ (see [9]). We denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$. As in [5], we can construct, for any $n \in \mathbb{N}$, the algebra

$$\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A})) = (P^{(n)}(\mathfrak{P}^*(\mathfrak{A})), (f_\gamma)_{\gamma<o(\tau)})$$

of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$. Some connections between the multialgebra $\mathfrak{A}$ and the term functions from $P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ are presented in [1].

Remark 1. [5, Corollary 8.2] For any $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $m \in \mathbb{N}$, $m \geq n$ there exists $q \in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

$$p(A_0, \ldots, A_{n-1}) = q(A_0, \ldots, A_{m-1})$$

for any $A_0, \ldots, A_{m-1} \in P^*(A)$.

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Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. The n-ary (strong) identity $\mathbf{q} = \mathbf{r}$ is said to be satisfied on a multialgebra $\mathfrak{A}$ if $q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1})$ for all $a_0, \ldots, a_{n-1} \in A$, where q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(\mathfrak{A})$. We can also consider that a weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is said to be satisfied on a multialgebra $\mathfrak{A}$ if $q(a_0, \ldots, a_{n-1}) \cap r(a_0, \ldots, a_{n-1}) \neq \emptyset$ for all $a_0, \ldots, a_{n-1} \in A$ (q and r have the same signification as before).

Remark 2. [8, Remark 2] A semihypergroup is a hypergroupoid $(H, \circ)$ for which the multioperation is associative. A semihypergroup $(H, \circ)$ (with $H \neq \emptyset$) is a hypergroup if and only if there exist two binary multioperations $/, \backslash$ on H such that on the multialgebra $(H, \circ, /, \backslash)$ are satisfied the following weak identities:

$$\mathbf{x}_1 \cap \mathbf{x}_0 \circ (\mathbf{x}_0 \backslash \mathbf{x}_1) \neq \emptyset, \ \mathbf{x}_1 \cap (\mathbf{x}_1 / \mathbf{x}_0) \circ \mathbf{x}_0 \neq \emptyset.$$

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \ldots, a_{n_\gamma - 1} \in A$ we have

$$h(f_\gamma(a_0, \ldots, a_{n_\gamma - 1})) \subseteq f_\gamma(h(a_0), \ldots, h(a_{n_\gamma - 1})). \tag{1}$$

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. The multialgebra isomorphisms can also be characterized as being those bijective homomorphisms for which (1) holds with equality.

Proposition 1. [7, Proposition 1] *For a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$ then $h(p(a_0, \ldots, a_{n-1})) \subseteq p(h(a_0), \ldots, h(a_{n-1}))$.*

The *fundamental relation* of a multialgebra $\mathfrak{A}$ is the transitive closure α^* of the relation α given on A as follows: for $x, y \in A$, $x \alpha y$ if and only if $x, y \in p(a_0, \ldots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \ldots, a_{n-1} \in A$ (see [6] and [8]). The relation α^* is the smallest equivalence relation on A such that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra. We denoted the class $\alpha^*\langle a \rangle$ of $a \in A$ modulo α^* by $\overline{a}$ and A/α^* by $\overline{A}$. We also denoted the algebra $\mathfrak{A}/\alpha^*$ by $\overline{\mathfrak{A}}$ and we called it the *fundamental algebra* of the multialgebra $\mathfrak{A}$.

Proposition 2. [8, Proposition 3] *The following conditions are equivalent for a multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type τ :*

(i) *for all $\gamma < o(\tau)$, for all $a_0, \ldots, a_{n_\gamma - 1} \in A$,*

$$a \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) \ \Rightarrow \ \overline{a} = f_\gamma(a_0, \ldots, a_{n_\gamma - 1}).$$

(ii) *for all $m \in \mathbb{N}$, for all $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, m-1\}\}$, for all $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in A$,*

$$q(a_0, \ldots, a_{m-1}) \cap r(b_0, \ldots, b_{m-1}) \neq \emptyset \ \Rightarrow \ q(a_0, \ldots, a_{m-1}) = r(b_0, \ldots, b_{m-1}).$$

Remark 3. From Remark 1 it follows that in condition (ii) is not necessary to consider that the arities of $\mathbf{q}$ and $\mathbf{r}$ are equal.

The multialgebras which verify one of the equivalent conditions from the previous proposition are generalizations for the complete (semi)hypergroups (see [3, Definition 137]). This fact suggested the following:

Definition 1. A multialgebra which satisfies one of the equivalent conditions from the previous proposition will be called *complete multialgebra.*

Proposition 3. [8, Proposition 4] *Let* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *be a multialgebra of type* τ. *The multialgebra* $\mathfrak{A}$ *is complete if and only if there exist a universal algebra* $\mathfrak{B} = (B, (f'_\gamma)_{\gamma<o(\tau)})$ *and a partition* $\{A_b \mid b \in B\}$ *of* A *such that* $A_{b_1} \cap A_{b_2} = \emptyset$ *for any* $b_1 \neq b_2$ *from* B *and for any* $\gamma < o(\tau)$ *and* $a_0, \ldots, a_{n_\gamma-1} \in A$ *with* $a_i \in A_{b_i}$ $(i \in \{0, \ldots, n_\gamma - 1\})$, *we have*

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = A_{f'_\gamma(b_0,\ldots,b_{n_\gamma-1})}. \tag{2}$$

Remark 4. For any $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \ldots, n-1\}\}$ and $a_0, \ldots, a_{n-1} \in A$, if $b_0, \ldots, b_{n-1} \in B$ such that $a_i \in A_{b_i}$ for each $i \in \{0, \ldots, n-1\}$, then

$$p(a_0, \ldots, a_{n-1}) = A_{p'(b_0,\ldots,b_{n-1})} \tag{3}$$

(p and p' denote the term functions induced by $\mathbf{p}$ on $\mathfrak{P}^*(\mathfrak{A})$ and $\mathfrak{B}$, respectively).

Remark 5. The fundamental relation of the multialgebra $\mathfrak{A}$ from Proposition 3 is the relation $\alpha^*_{\mathfrak{A}} = \alpha_{\mathfrak{A}}$ defined by $x \alpha_{\mathfrak{A}} y$ if and only if

$$x = y \in A \setminus \left(\bigcup \{A_{f'_\gamma(b_0,\ldots,b_{n_\gamma-1})} \mid \gamma < o(\tau),\ b_0, \ldots, b_{n_\gamma-1} \in B\} \right)$$

or $x, y \in A_{f'_\gamma(b_0,\ldots,b_{n_\gamma-1})}$ for some $\gamma < o(\tau)$ and $b_0, \ldots, b_{n_\gamma-1} \in B$.

The main result of this paper is the following theorem:

Theorem 1. *A multialgebra* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *is complete if and only if there exists a structure of universal algebra* $\mathfrak{A}'' = (A, (f''_\gamma)_{\gamma<o(\tau)})$ *on* A *and a congruence relation* ρ *on* $\mathfrak{A}''$ *such that for each* $\gamma < o(\tau)$ *and for any* $a_0, \ldots, a_{n_\gamma-1} \in A$,

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \rho\langle f''_\gamma(a_0, \ldots, a_{n_\gamma-1})\rangle. \tag{4}$$

Proof. Let $\mathfrak{A}'' = (A, (f''_\gamma)_{\gamma<o(\tau)})$ be a universal algebra and let $\rho \subseteq A \times A$ be a congruence relation on $\mathfrak{A}''$. The fact that the multialgebra $\mathfrak{A}$ defined by (4) is complete follows by considering in Proposition 3, $B = A/\rho$, $\mathfrak{B} = \mathfrak{A}''/\rho$ and $A_{\rho\langle a\rangle} = \rho\langle a\rangle$ for any $\rho\langle a\rangle \in B$.

Conversely, let us consider that the multialgebra $\mathfrak{A}$ is complete and let $\mathfrak{B}$ and $\{A_b \mid b \in B\}$ be as in Proposition 3. Let us choose in each A_b an element a_{A_b}. For any $\gamma < o(\tau)$ and for any $a_0, \ldots, a_{n_\gamma-1} \in A$ there exist $b_0, \ldots, b_{n_\gamma-1} \in B$ (uniquely determined) such that $a_0 \in A_{b_0}, \ldots, a_{n_\gamma-1} \in A_{b_{n_\gamma-1}}$. If we define

$$f''_\gamma(a_0, \ldots, a_{n_\gamma-1}) = a_{A_{f_\gamma(b_0,\ldots,b_{n_\gamma-1})}}$$

then we obtain a universal algebra $\mathfrak{A}''$ on A. The relation $\rho = \bigcup_{b\in B} A_b \times A_b$ is a congruence on $\mathfrak{A}''$ and (4) holds. □

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a universal algebra. It is clear that

$$A_* = \{f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid a_0, \ldots, a_{n_\gamma-1} \in A,\ \gamma < o(\tau)\}$$

is a subalgebra of $\mathfrak{A}$. Let us denote by $\mathfrak{A}_*$ the universal algebra determined on A_* by the restrictions of the operations $(f_\gamma)_{\gamma<o(\tau)}$. If we consider an equivalence relation ρ on A and for any $\gamma < o(\tau)$ and $a_0, \ldots, a_{n_\gamma-1} \in A$ we take

$$f^\rho_\gamma(a_0, \ldots, a_{n_\gamma-1}) = \rho\langle f_\gamma(a_0, \ldots, a_{n_\gamma-1})\rangle \tag{5}$$

we obtain a multialgebra $\mathfrak{A}_\rho = (A, (f^\rho_\gamma)_{\gamma<o(\tau)})$ on A.

From Theorem 1 we deduce that the request for ρ to be a congruence on $\mathfrak{A}$ is sufficient for $\mathfrak{A}_\rho$ to be a complete multialgebra. A trivial example will show that this condition is not necessary.

Example 1. It is clear that any universal algebra is a complete multialgebra. Let us consider the groupoid $(H,\cdot)$ given by the following table:

$\cdot$	a	b	c	d
a	c	d	c	d
b	d	c	c	d
c	c	c	c	d
d	d	d	d	d

and the relation $\rho=(\{a,b\}\times\{a,b\})\cup(\{c\}\times\{c\})\cup(\{d\}\times\{d\})$. Since $(a,b)\in\rho$, $ab=d$, $bb=c$ and $(c,d)\notin\rho$, the relation ρ is not a congruence on $(H,\cdot)$, but if we consider $x\circ y=\rho\langle xy\rangle$ for any $x,y\in H$ we obtain a groupoid $(H,\circ)=(H,\cdot)$ which is, obviously, a complete multialgebra.

A necessary and sufficient condition on ρ such that $\mathfrak{A}_\rho$ is a complete multialgebra will be introduced in the following theorem.

Theorem 2. *The multialgebra $\mathfrak{A}_\rho=(A,(f_\gamma^\rho)_{\gamma<o(\tau)})$ is complete if and only if*

$$\rho'=\left(\bigcup_{a\in A\setminus\rho(A_*)}\{a\}\times\{a\}\right)\cup(\rho\cap(\rho(A_*)\times\rho(A_*)))$$

is a congruence relation on $\mathfrak{A}$.

Proof. It is easy to observe that ρ' is an equivalence relation on A and $\mathfrak{A}_\rho=\mathfrak{A}_{\rho'}$ thus the assumption that ρ' is a congruence relation on $\mathfrak{A}$ leads us to the conclusion that the multialgebra $\mathfrak{A}_\rho$ is complete.

Conversely, let us consider that $\mathfrak{A}_\rho$ is a complete multialgebra and let us prove that ρ' is a congruence relation on $\mathfrak{A}$. It is enough to prove that for any $\gamma<o(\tau)$ and $a_0,\dots,a_{n_\gamma-1},x,y\in A$ with $x\rho' y$ we have

$$f_\gamma(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})\rho' f_\gamma(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1}). \tag{6}$$

Since $f_\gamma(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})$, $f_\gamma(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1})\in A_*$, (6) can be written again as

$$f_\gamma(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})\rho f_\gamma(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1}). \tag{6'}$$

If $x=y\in A\setminus\rho(A_*)$ then $(6')$ holds trivially.
If $x,y\in\rho(A_*)$ then there exist $\delta,\zeta<o(\tau)$ and $x_0,\dots,x_{n_\delta-1}$, $y_0,\dots,y_{n_\zeta-1}\in A$ with

$$x\rho f_\delta(x_0,\dots,x_{n_\delta-1}) \text{ and } y\rho f_\zeta(y_0,\dots,y_{n_\zeta-1}).$$

Using (5) it follows that for any $i\in\{0,\dots,n_\gamma-1\}$ we have

$$f_\delta^\rho(x_0,\dots,x_{n_\delta-1})=\rho\langle x\rangle=\rho\langle y\rangle=f_\zeta^\rho(y_0,\dots,y_{n_\zeta-1}). \tag{7}$$

The nonempty set $f_\gamma^\rho(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})$ is a subset for

$$f_\gamma^\rho(a_0,\dots,a_{i-1},f_\delta^\rho(x_0,\dots,x_{n_\delta-1}),a_{i+1},\dots,a_{n_\gamma-1}).$$

Let $m=n_\delta+n_\gamma$ and let $b_0,\dots,b_{m-1}$ be $a_0,\dots,a_{n_\gamma-1}$, $x_0,\dots,x_{n_\delta-1}$ respectively. According to Remark 1, there exists $p^\rho\in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}_\rho))$ such that

$$f_\gamma^\rho(a_0,\dots,a_{i-1},f_\delta^\rho(x_0,\dots,x_{n_\delta-1}),a_{i+1},\dots,a_{n_\gamma-1})=p^\rho(b_0,\dots,b_{m-1}).$$

Since $\mathfrak{A}_\rho$ is a complete multialgebra and $f_\gamma^\rho(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})$ is included in $p^\rho(b_0,\dots,b_{m-1})$ we have

$$f_\gamma^\rho(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})=p^\rho(b_0,\dots,b_{m-1}).$$

Using (7) we obtain that $f_\gamma^\rho(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1})$ is a subset of

$$\begin{aligned}&f_\gamma^\rho(a_0,\dots,a_{i-1},f_\zeta^\rho(y_0,\dots,y_{n_\zeta-1}),a_{i+1},\dots,a_{n_\gamma-1})\\ =&f_\gamma^\rho(a_0,\dots,a_{i-1},f_\delta^\rho(x_0,\dots,x_{n_\delta-1}),a_{i+1},\dots,a_{n_\gamma-1})\\ =&p^\rho(b_0,\dots,b_{m-1})\end{aligned}$$

hence $f_\gamma^\rho(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1})=p^\rho(b_0,\dots,b_{m-1})$. So,

$$f_\gamma^\rho(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})=f_\gamma^\rho(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1}).$$

Thus

$$\rho\langle f_\gamma(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_{n_\gamma-1})\rangle=\rho\langle f_\gamma(a_0,\dots,a_{i-1},y,a_{i+1},\dots,a_{n_\gamma-1})\rangle$$

and $(6')$ holds. □

Corollary 1. *Let $\mathfrak{A}$ be a universal algebra and let ρ be an equivalence relation on A such that $\mathfrak{A}_\rho$ is a complete multialgebra. For any $n\in\mathbb{N}$, $\mathbf{p}\in\mathbf{P}^{(n)}(\tau)\setminus\{\mathbf{x}_i\mid i\in\{0,\dots,n-1\}\}$ and $a_0,\dots,a_{n-1}\in A$ we have*

$$p^\rho(a_0,\dots,a_{n-1})=\rho\langle p(a_0,\dots,a_{n-1})\rangle$$

(p^ρ and p denote the term functions induced by $\mathbf{p}$ on $\mathfrak{P}^*(\mathfrak{A}_\rho)$ and $\mathfrak{A}$, respectively).

Indeed, taking in Remark 4, $\mathfrak{B}=\mathfrak{A}/\rho'$ and $A_{\rho'\langle a\rangle}=\rho'\langle a\rangle$ for any $a\in A$ we have

$$p^\rho(a_0,\dots,a_{n-1})=\rho'\langle p(a_0,\dots,a_{n-1})\rangle=\rho\langle p(a_0,\dots,a_{n-1})\rangle.$$

Corollary 2. *Let $\mathfrak{A}$ be a universal algebra which verifies the identity $\mathbf{q}=\mathbf{r}$ ($\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)$) and let ρ be an equivalence relation on A such that $\mathfrak{A}_\rho$ is a complete multialgebra.*

i) *If $\mathbf{q}=\mathbf{x}_i$ and $\mathbf{r}=\mathbf{x}_j$ for some $i,j\in\{0,\dots,n-1\}$, $i\neq j$ then $|A|=1$ and the identity $\mathbf{q}=\mathbf{r}$ is trivially satisfied on $\mathfrak{A}_\rho$.*
ii) *If $\mathbf{q}=\mathbf{x}_i$ for some $i\in\{0,\dots,n-1\}$ and $\mathbf{r}\in\mathbf{P}^{(n)}(\tau)\setminus\{\mathbf{x}_i\mid i\in\{0,\dots,n-1\}\}$ then the identity $\mathbf{q}\cap\mathbf{r}\neq\emptyset$ is satisfied on $\mathfrak{A}_\rho$.*
iii) *If $\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)\setminus\{\mathbf{x}_i\mid i\in\{0,\dots,n-1\}\}$ and the identity $\mathbf{q}=\mathbf{r}$ is satisfied on $\mathfrak{A}$ then the identity $\mathbf{q}=\mathbf{r}$ is satisfied on $\mathfrak{A}_\rho$.*

From Remark 5 we deduce:

Corollary 3. *If $\mathfrak{A}$ is a universal algebra and ρ is an equivalence relation on A such that $\mathfrak{A}_\rho$ is a complete multialgebra then the fundamental relation of $\mathfrak{A}_\rho$ is*

$$\alpha^*_{\mathfrak{A}}=\alpha_{\mathfrak{A}}=\left(\bigcup_{a\in A\setminus\rho(A_*)}\{a\}\times\{a\}\right)\cup\left(\rho\cap\left(\rho(A_*)\times\rho(A_*)\right)\right).$$

It is known that any group $(G,\cdot)$ can be seen as a universal algebra with three binary operations $(G,\cdot,/,\backslash)$, with $G\neq\emptyset$, which satisfies the following identities

$$(\mathbf{x}_0\cdot\mathbf{x}_1)\cdot\mathbf{x}_2=\mathbf{x}_0\cdot(\mathbf{x}_1\cdot\mathbf{x}_2),\ \mathbf{x}_1=\mathbf{x}_0\cdot(\mathbf{x}_0\backslash\mathbf{x}_1),\ \mathbf{x}_1=(\mathbf{x}_1/\mathbf{x}_0)\cdot\mathbf{x}_0,$$

$$\mathbf{x}_1=\mathbf{x}_0\backslash(\mathbf{x}_0\cdot\mathbf{x}_1),\ \mathbf{x}_1=(\mathbf{x}_1\cdot\mathbf{x}_0)/\mathbf{x}_0$$

(see [10, p.215]). Using the previous notations we have $G_*=G$ and for any equivalence relation ρ on G we have $\rho(G_*)=G$.

From Remark 2, Theorem 2, Corollary 2 and Corollary 3 we obtain:

Corollary 4. *Let $(G,\cdot)$ be a group and let ρ be an equivalence relation on G. The hypergroupoid $(G,\circ)$, given by $x\circ y=\rho\langle xy\rangle$, is a complete multialgebra if and only if there exists a normal subgroup N of G such that ρ is the equivalence relation induced by N on G. In this case $x\circ y=(xy)N$ and $(G,\circ)$ is a complete hypergroup with the fundamental relation $\beta=\bigcup_{g\in G} gN\times gN$. The fundamental group of the hypergroup $(G,\circ)$ is the factor group $(G/N,\cdot)$ and the heart of $(G,\circ)$ is $\beta\langle 1\rangle=N$.*

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