# DIRECT LIMITS OF DIRECT SYSTEMS OF HYPERGROUPOIDS ASSOCIATED TO BINARY RELATIONS

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Abstract. Using some previous results on hypergroupoids associated to binary relations, we will point out some isomorphisms between categories of relational structures with one binary relation and categories of hypergroupoids (or semihypergroups, or hypergroups). This will lead us to some interesting results on direct limits of hypergroupoids (or semihypergroups, or hypergroups) associated to binary relation.

## 1. Introduction

In [8], Rosenberg associates to each binary homogeneous relation with full domain $R$ on a set $H$ a hypergroupoid $H_R$. In the same paper, Rosenberg establishes necessary and sufficient conditions for $R$ such that $H_R$ is a semihypergroup (or a hypergroup) and necessary and sufficient conditions for a (semi)hypergroup to be the (semi)hypergroup determined by a binary relation. Later, in [1], Corsini studies when some constructions in the class of relational structures above provides a hypergroup through the association rule introduced by Rosenberg. An extended view on this topic is found in [2]. Yet, a categorical translation for some results from [8] and [1] seems to enlarge the existing frame. The categorical tools are not very difficult and they can be found in [4] or [6]. We begin with a categorical survey of the subject, and then, using the results obtained in [5], we will be able to strengthen some of the results presented by Corsini in [1] on direct limits of direct systems.

## 2. Preliminaries

Let $H$ be a set and let $R$ be a binary relation on $H$. Denote the inverse of the relation $R$ by $\overset{-1}{R}$. As in [8], we call the set

$$D(R) = \{x \in H \mid \exists y \in H : xRy\}$$

the domain of the relation $R$. For $x \in H$, $X \subseteq H$ we denote

$$R\langle x\rangle = \{y \in H \mid xRy\} \text{ and } R(X) = \{y \in H \mid \exists x \in X : xRy\}.$$

If $x_1, \dots, x_n \in H$, we will write $R(x_1, \dots, x_n)$ instead of $R(\{x_1, \dots, x_n\})$. Clearly,

$$D(R) = \overset{-1}{R}(H) \text{ and } R(X) = \bigcup_{x \in X} R\langle x\rangle.$$

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As in [8], one can associate to the binary relation $R \subseteq H \times H$ the partial hypergroupoid $H_R = (H, \circ)$ defined by

$$x \circ y = R(x, y). \tag{1}$$

It is obvious that

$$x^2 = x \circ x = R\langle x \rangle = \{y \in H \mid xRy\} \text{ and } x \circ y = x^2 \cup y^2.$$

**Lemma 1.** [8, Lemma 1] *Let $H$ be a set and let $R$ be a binary relation on $H$. The partial hypergroupoid $H_R = (H, \circ)$ is a hypergroupoid if and only if $R$ has a full domain (i.e. the domain of $R$ is $H$).*

An element $x \in H$ is an *outer element* of (the relation) $R$ if there exists $h \in H$ such that $(h, x) \notin R^2$.

**Proposition 1.** [8, Proposition 2] *Let $R$ be a binary relation on $H$ with full domain. The hypergroupoid $H_R$ is a semihypergroup if and only if $R \subseteq R^2$ and*

$$(a, x) \in R^2 \Rightarrow (a, x) \in R$$

*whenever $x$ is an outer element of $R$.*

**Proposition 2.** [8] *Let $H \neq \emptyset$ and let $R$ be a binary relation on $H$. The hypergroupoid $H_R$ is a hypergroup if and only if the following conditions hold:*

1) $\overset{-1}{R}(H) = H$;
2) $R(H) = H$;
3) $R \subseteq R^2$;
4) *whenever $x$ is an outer element of $R$ we have*

$$(a, x) \in R^2 \Rightarrow (a, x) \in R.$$

*Remark* 1. For any reflexive relation $R$ on $H$ the partial hypergroupoid $H_R = (H, \circ)$ is a hypergroupoid. If, in addition, $R$ is transitive and $H \neq \emptyset$ the conditions 2), 3) and 4) from the above theorem are also satisfied, so $H_R = (H, \circ)$ is a hypergroup.

Let $(H, R)$, $(H', R')$ be relational systems with binary relations and $h : H \to H'$. One says that $h$ is an *homomorphism* of relational systems if

$$xRy \Rightarrow h(x)R'h(y).$$

Let $(H, \circ)$, $(H', \circ')$ be hypergroupoids. A mapping $h : H \to H'$ is called *homomorphism* (of hypergroupoids) if

$$h(x \circ y) \subseteq h(x) \circ' h(y),\ \forall x, y \in H.$$

*Remark* 2. If $H$ is a set and $R$ is a binary relation on $H$ with full domain, we can identify $(H, R)$ with the multialgebra $(H, f)$ with one unary multioperation $f : H \to P^*(H)$ defined by

$$xRy \Leftrightarrow y \in f(x). \tag{2}$$

If $(H', R')$ is also a relational system for which $\overset{-1}{R'}(H') = H'$, $(H', f')$ is the corresponding monounary multialgebra and $h : H \to H'$ is a relational homomorphisms between $(H, R)$ and $(H', R')$ then:

$$[xRy \Rightarrow h(x)R'h(y)] \Leftrightarrow [y \in f(x) \Rightarrow h(y) \in f'(h(x))] \Leftrightarrow h(f(x)) \subseteq f'(h(x)).$$

This means that $h$ is a relational homomorphism between $(H, R)$ and $(H', R')$ if and only if $h$ is a homomorphism between the multialgebras $(H, f)$ and $(H', f')$.

Let $\mathcal{R}_2$ be the category of the relational systems with one binary relation – the morphisms are the homomorphisms of relational systems and the product of two morphisms is the usual composition of homomorphisms – and let us denote by $\mathcal{R}_2'$ the full subcategory of $\mathcal{R}_2$ whose objects are the relational systems $(H, R)$ for which $\overset{-1}{R}(H) = H$. From the above remark it follows that the category $\mathcal{R}_2'$ is isomorphic to the category **Malg**(1) of the monounary multialgebras (here the morphisms are the multialgebra homomorphisms and the product of two morphisms is the usual composition of homomorphisms).

The hypergroupoids (or semihypergroups, or hypergroups) associated to binary relations can be seen as hypergroupoids (or semihypergroups, or hypergroups) associated to monounary multialgebras $(H, f)$ using the translation of (1) in the terms of the unary multioperation $f$. Remember that

$$f(X) = \bigcup_{x\in X} f(x),\ \forall X \subseteq H,\ X \neq \emptyset,$$

and (1) becomes

$$x \circ y = f(\{x, y\}) = f(x) \cup f(y)(= x^2 \cup y^2).$$

Lema 1 can be rewritten as below:

**Lemma 2.** *For any multialgebra $(H, f)$ with one unary multioperation, the equality*

$$x \circ y = f(\{x, y\})$$

*defines a hypergroupoid $H_f = (H, \circ)$.*

Let $f$ and $R$ be as in (2). An $x \in H$ is an *outer element* (of $(H, f)$) if there exists $h \in H$ such that $x \notin f(f(h))$. Now, Propositions 1 and 2 can be restated as follows:

**Proposition 3.** *Let $(H, f)$ be a multialgebra with one unary multioperation. The hypergroupoid $H_f$ is a semihypergroup if and only if*

$$f(x) \subseteq f(f(x)),\ \forall x \in H$$

*and for any outer element $x \in H$,*

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

**Proposition 4.** *Let $H \neq \emptyset$ and let $(H, f)$ be a multialgebra with one unary multioperation. The hypergroupoid $H_f$ is a hypergroup if and only if the following conditions hold:*

i) *$f(H) = H$;*
ii) *$f(x) \subseteq f(f(x)),\ \forall x \in H$;*
iii) *whenever $x$ is an outer element we have*

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

## 3. A categorical survey of the subject

In [8], Rosenberg determines the semihypergroups which can be obtained from a binary relations using (1):

**Proposition 5.** [8, Proposition 3] *Let $(H, *)$ be a semihypergroup. There exists a binary relation $R$ on $H$ such that $(H, *) = H_R$ if and only if the following conditions are satisfied for any $x, y \in H$:*

a) $x * y = x^2 \cup y^2$;
b) $x^2 \subseteq (x^2)^2$;
c) $(x^2)^2 \cap (H \setminus (y^2)^2) \subseteq x^2$.

The binary relation $R \subseteq H \times H$ from Rosenberg's proof is defined by

$$xRy \Leftrightarrow y \in x * x$$

and $\overset{-1}{R}(H) = H$. It follows easily that Proposition 5 can be restated as bellow.

**Proposition 6.** *Let $(H, *)$ be a hypergroupoid. There exists a binary relation $R$ on $H$ such that $(H, *) = H_R$ if and only if*

$$x * y = x^2 \cup y^2, \ \forall x, y \in H. \tag{3}$$

*A hypergroupoid $(H, *)$ which satisfies the condition* (3) *is a semihypergroup if and only if it verifies the conditions* b) *and* c) *from Proposition 5.*

*Remark* 3. A hypergroupoid $(H, *)$ which satisfies the condition (3) is a hypergroup if and only if it verifies the conditions b), c) from Proposition 5, and $\bigcup_{x \in H} x^2 = H$.

*Remark* 4. In the terms of our discussion, a hypergroupoid (or semihypergroup, or hypergroup) $(H, *)$ is determined by a unary multioperation $f$ on $H$ if and only if $(H, *)$ satisfies the condition (3).

Besides $\mathcal{R}_2$, $\mathcal{R}'_2$ and $\mathbf{Malg}(1)$, the following categories drew our attention:
• the category $\mathbf{Malg}(2)$ of hypergroupoids: the morphisms are the hypergroupoid homomorphisms and the product of two morphisms is the usual composition of homomorphisms;
• the full subcategory of $\mathbf{Malg}(2)$ whose object are the hypergroupoids which satisfy (3), denoted by $\mathbf{Malg}'(2)$;
• the full subcategory of $\mathbf{Malg}(2)$ whose object are the semihypergroups, denoted by $\mathbf{SHG}$;
• the full subcategory of $\mathbf{SHG}$ whose object are the semihypergroups which satisfy (3), denoted by $\mathbf{SHG}'$;
• the category $\mathbf{HG}$ of hypergroups with hypergroup homomorphisms and the usual composition;
• the full subcategory of $\mathbf{HG}$ whose object are the hypergroups which satisfy (3), denoted by $\mathbf{HG}'$;
• the full subcategory $\mathbf{Malg}'(1)$ of $\mathbf{Malg}(1)$ whose objects are the monounary multialgebras $(H, f)$ which satisfy the conditions from Proposition 3 (i.e. the conditions ii),iii) from Proposition 4);
• the full subcategory $\mathbf{Malg}''(1)$ of $\mathbf{Malg}(1)$ whose objects are the monounary multialgebras $(H, f)$, with $H \neq \emptyset$, which satisfy the conditions i), ii), iii) from Proposition 4.

The correspondence $(H, f) \mapsto H_f = (H, \circ)$ defines, in a natural way, a functor from $\mathbf{Malg}(1)$ into $\mathbf{Malg}(2)$.

**Lemma 3.** *Let $(H, f)$, $(H', f')$ be two multialgebras from $\mathbf{Malg}(1)$, let $H_f = (H, \circ)$, $H'_{f'} = (H', \circ')$ be the hypergroupoids associated to the multioperations $f$ and $f'$, respectively. The mapping $h : H \to H'$ is a homomorphism from $(H, f)$ into $(H', f')$ if and only if $h$ is a homomorphism from $H_f$ into $H'_{f'}$.*

*Proof.* If $h$ is a homomorphism from $(H, f)$ into $(H', f')$ then, for any $x, y \in H$,

$$h(x \circ y) = h(f(x) \cup f(y)) = h(f(x)) \cup h(f(y)) \subseteq f'(h(x)) \cup f'(h(y)) = h(x) \circ' h(y),$$

hence $h$ is a homomorphism from $H_f$ into $H'_{f'}$. Conversely, if $h$ is a homomorphism from $H_f$ into $H'_{f'}$ and $x \in H$ then

$$h(f(x)) = h(x \circ x) \subseteq h(x) \circ h(x) = f'(h(x)),$$

thus $h$ is a multialgebra homomorphism from $(H, f)$ into $(H', f')$. □

**Corollary 1.** *Let us consider the correspondences*

$$(H, f) \mapsto H_f$$

*for any object* $(H, f)$ *from* $\mathbf{Malg}(1)$ *and*

$$h \mapsto h$$

*for any morphism* $h \in H_{\mathbf{Malg}(1)}((H, f), (H', f'))$. *We obtain a covariant functor*

$$F : \mathbf{Malg}(1) \to \mathbf{Malg}'(2).$$

*Remark* 5. If in Lemma 3 we consider two multialgebras $(H, f)$ and $(H', f')$ from $\mathbf{Malg}'(1)$ then $h$ is a morphism in $\mathbf{SHG}'$ between $H_f$ and $H'_{f'}$. Hence, we can define as in Corollary 1 a covariant functor

$$F' : \mathbf{Malg}'(1) \to \mathbf{SHG}'.$$

Also, if in Lemma 3 we consider two multialgebras $(H, f)$ and $(H', f')$ from $\mathbf{Malg}''(1)$ then $h$ is a morphism in $\mathbf{HG}'$ between $H_f$ and $H'_{f'}$. Hence, we can define as in Corollary 1 a covariant functor

$$F'' : \mathbf{Malg}''(1) \to \mathbf{HG}'.$$

*Remark* 6. Let $(H, *)$ be a hypergroupoid and consider the multioperation

$$f_* : H \to P^*(H), \; f_*(x) = x * x.$$

Then $(H, f_*)$ is in $\mathbf{Malg}(1)$, and, as in the "only if" part of Lemma 3, we deduce that if $h \in H_{\mathbf{Malg}(2)}((H, *), (H', *'))$ is a morphism in $\mathbf{Malg}(2)$ then $h$ is a multialgebra homomorphism from $(H, f_*)$ into $(H', f_{*'})$, i.e. $h \in H_{\mathbf{Malg}(1)}((H, f_*), (H', f_{*'}))$. It follows immediately that the correspondences

$$(H, *) \mapsto (H, f_*), \; h \mapsto h$$

define a covariant functor $\mathbf{Malg}(2) \to \mathbf{Malg}(1)$. We compose this functor with the inclusion functor $\mathbf{Malg}'(2) \to \mathbf{Malg}(2)$ and we obtain a covariant functor

$$G : \mathbf{Malg}'(2) \to \mathbf{Malg}(1).$$

**Lemma 4.** *The covariant functor $F$ is an isomorphism between the categories* $\mathbf{Malg}(1)$ *and* $\mathbf{Malg}'(2)$, *and $G$ is the inverse of $F$.*

*Proof.* Let $1_{\mathbf{Malg}(1)}$ and $1_{\mathbf{Malg}'(2)}$ denote the identity functors of $\mathbf{Malg}(1)$ and $\mathbf{Malg}'(2)$, respectively. We show that

$$GF = 1_{\mathbf{Malg}(1)} \text{ and } FG = 1_{\mathbf{Malg}'(2)}.$$

Let $(H, f) \in \mathbf{Malg}(1)$ and $H_f = (H, \circ)$. We have

$$G(F(H, f)) = G(H, \circ) = (H, f_\circ)$$

and $(H, f) = (H, f_\circ)$ since for any $x \in H$,

$$f_\circ(x) = x \circ x = f(x) \cup f(x) = f(x).$$

If $h \in H_{\mathbf{Malg}(1)}((H, f), (H', f'))$ is a multialgebra homomorphism then

$$G(F(h)) = G(h) = h.$$

Let $(H, *) \in \mathbf{Malg}'(2)$. We have

$$F(G(H, *)) = F(H, f_*) = H_{f_*} = (H, \circ),$$

and for any $x, y \in H$,

$$x \circ y = f_*(x) \cup f_*(y) = (x * x) \cup (y * y) = x * y,$$

thus $(H, \circ) = (H, *)$ and if $h \in H_{\mathbf{Malg}'(2)}((H, *), (H', *'))$ then

$$G(F(h)) = G(h) = h$$

which ends the proof of the lemma. □

**Corollary 2.** *The functors $F$ and $G$ provide an isomorphism between the categories $\mathcal{R}_2'$ and $\mathbf{Malg}'(2)$.*

**Corollary 3.** *The functor $F'$ is an isomorphism between the categories $\mathbf{Malg}'(1)$ and $\mathbf{SHG}'$, and the inverse of $F'$ is the functor $G' : \mathbf{SHG}' \to \mathbf{Malg}'(1)$ given by*

$$G'(H, *) = (H, f_*),\ G'(h) = h.$$

**Corollary 4.** *The functor $F''$ is an isomorphism between the categories $\mathbf{Malg}''(1)$ and $\mathbf{HG}'$, and the inverse of $F''$ is the functor $G'' : \mathbf{HG}' \to \mathbf{Malg}''(1)$ given by*

$$G''(H, *) = (H, f_*),\ G''(h) = h.$$

## 4. Direct limits of direct systems of hypergroupoids associated to binary relations

Let $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ be a direct system of multialgebras from $\mathbf{Malg}(1)$ and for each $i \in I$ let $F(H_i, f_i) = (H_i, \circ_i)$. Clearly, $((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ is a direct system of hypergroupoids. We denote it by $F(\mathcal{H})$.

Remember that $(I, \leq)$ is a directed preordered set and the homomorphisms $\varphi_{ij}$ $(i, j \in I,\ i \leq j)$ are such that

$$\varphi_{ii} = 1_{A_i},\ \forall i \in I \text{ and } \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik},\ \forall i, j, k \in I,\ i \leq j \leq k.$$

The relation $\equiv$ defined on the disjoint union $H$ of the sets $H_i$ as follows: for any $x, y \in A$ there exist $i, j \in I$ such that $x \in H_i,\ y \in H_j$, and

$$x \equiv y \Leftrightarrow \exists k \in I,\ i \leq k,\ j \leq k :\ \varphi_{ik}(x) = \varphi_{jk}(y)$$

is an equivalence relation on $H$ and the factor set $H_\infty = H/_\equiv = \{\widehat{x} \mid x \in H\}$ is the direct limit of the direct system of sets $((H_i \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \leq j))$ (see [3]).

The direct limit $\varinjlim \mathcal{H}$ of the direct system of multialgebras multialgebras $\mathcal{H}$ is the monounary multialgebra $(H_\infty, f)$ with $f$ defined as follows: if $\widehat{x} \in A_\infty$ and $i \in I$ such that $x \in H_i$ then

$$f(\widehat{x}) = \{\widehat{y} \mid \exists m \in I,\ i \leq m,\ y \in f_m(\varphi_{im}(x))\}. \tag{2}$$

The direct limit of the direct system of hypergroupoids $((H_i, \circ_i) \mid i \in I)$ (in **Malg**(2)) is the hypergroupoid $(H_\infty, \circ)$ with $\circ$ defined as follows: if $\widehat{x_1}, \widehat{x_2} \in A_\infty$ and $i_1, i_2 \in I$ such that $x_1 \in H_{i_1}$, $x_2 \in H_{i_2}$ then

$$\widehat{x_1} \circ \widehat{x_2} = \{\widehat{y} \mid \exists m \in I,\ i_1 \le m,\ i_2 \le m,\ y \in \varphi_{i_1 m}(x_1) \circ_m \varphi_{i_2 m}(x_2)\} \tag{3}$$

(see [5, Lemma 15]). Using the above notations, we obtain the main result of this section.

**Theorem 1.** *The hypergroupoid $(H_\infty, \circ)$ is the hypergroupoid determined by the multialgebra $(H_\infty, f)$.*

*Proof.* Let $(H_\infty, *)$ be the hypergroupoid determined by $(H_\infty, f)$. We prove that

$$(H_\infty, *) = (H_\infty, \circ).$$

Let $\widehat{x_1}, \widehat{x_2} \in H_\infty$ with $x_1 \in H_{i_1}$, $x_2 \in H_{i_2}$ $(i_1, i_2 \in I)$. Let

$$\widehat{y} \in \widehat{x_1} * \widehat{x_2} = f(\widehat{x_1}) \cup f(\widehat{x_2}).$$

If $\widehat{y} \in f(\widehat{x_1})$ then there exists $m_1 \in I$, $i_1 \le m_1$ such that

$$y \equiv y' \text{ and } y' \in f_{m_1}(\varphi_{i_1 m_1}(x_1)).$$

If $\widehat{y} \in f(\widehat{x_2})$ then there exists $m_2 \in I$, $i_2 \le m_2$ such that

$$y \equiv y'' \in f_{m_2}(\varphi_{i_2 m_2}(x_2)).$$

Since $y' \equiv y \equiv y''$ there exists an upper bound $m \in I$ for $m_1$ and $m_2$ such that

$$\varphi_{m_1 m}(y') = \varphi_{m_2 m}(y'').$$

It follows immediately that

$$\begin{aligned} y &\equiv \varphi_{m_1 m}(y') \in \varphi_{m_1 m}(f_{m_1}(\varphi_{i_1 m_1}(x_1))) \subseteq f_m(\varphi_{i_1 m}(x_1)), \\ y &\equiv \varphi_{m_2 m}(y'') \in \varphi_{m_2 m}(f_{m_2}(\varphi_{i_2 m_2}(x_2))) \subseteq f_m(\varphi_{i_2 m}(x_2)). \end{aligned}$$

Thus $\varphi_{m_1 m}(y') = \varphi_{m_2 m}(y'')$ is a representative for $\widehat{y}$, and in $H_m$ we have

$$\varphi_{m_1 m}(y') \in f_m(\varphi_{i_1 m}(x_1)) \cup f_m(\varphi_{i_2 m}(x_2)) = \varphi_{i_1 m}(x_1) \circ_m \varphi_{i_2 m}(x_2).$$

From (3) it follows $\widehat{y} \in \widehat{x_1} \circ \widehat{x_2}$, hence we have proved the inclusion

$$\widehat{x_1} * \widehat{x_2} \subseteq \widehat{x_1} \circ \widehat{x_2}.$$

Conversely, let $\widehat{y} \in \widehat{x_1} \circ \widehat{x_2}$. According to (3) it follows that there exists an upper bound $m \in I$ for $i_1$ and $i_2$ such that in $H_m$ we have

$$y \in \varphi_{i_1 m}(x_1) \circ_m \varphi_{i_2 m}(x_2) = f_m(\varphi_{i_1 m}(x_1)) \cup f_m(\varphi_{i_2 m}(x_2)).$$

If $y \in f_m(\varphi_{i_1 m}(x_1))$ then $\widehat{y} \in f(\widehat{x_1})$ and if $y \in f_m(\varphi_{i_2 m}(x_2))$ then $\widehat{y} \in f(\widehat{x_2})$. Thus

$$\widehat{y} \in f(\widehat{x_1}) \cup f(\widehat{x_2}) = \widehat{x_1} * \widehat{x_2}$$

and we have proved the equality

$$\widehat{x_1} * \widehat{x_2} = \widehat{x_1} \circ \widehat{x_2}$$

for any $\widehat{x_1}, \widehat{x_2} \in H_\infty$. $\square$

Since $(H_\infty, f)$, with the homomorphisms $\varphi_{i\infty} : H_i \to H_\infty,\ \varphi_{i\infty}(x) = \widehat{x}\ (i \in I)$ is the direct limit in $\mathbf{Malg}(1)$ of the direct system $\mathcal{H}$ and $(H_\infty, \circ)$, with the homomorphisms $\varphi_{i\infty}$, is the direct limit in $\mathbf{Malg}(2)$ of the direct system of hypergroupoids $((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ (see [5]) it is quite clear now that the direct limit of a direct system of hypergroupoids which satisfy (3) is a hypergroupoid which satisfy (3). Thus we have:

**Corollary 5.** *The subcategory* $\mathbf{Malg}'(2)$ *of* $\mathbf{Malg}(2)$ *is closed under direct limits of direct systems.*

**Corollary 6.** *If* $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *is a direct system of multialgebras from* $\mathbf{Malg}(1)$ *and* $F(\mathcal{H})$ *is the direct system of hypergroupoids* $((F(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *then* $F(\varinjlim \mathcal{H})$ *is the direct limit of* $F(\mathcal{H})$ *in* $\mathbf{Malg}(2)$.

**Corollary 7.** *If* $((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *is a direct system of hypergroupoids from* $\mathbf{Malg}'(2)$ *then the direct limit of the direct system of multialgebras* $((G(H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *from* $\mathbf{Malg}(2)$ *is the monounary multialgebra which determines the direct limit of the hypergroupoids* $((H_i, \circ_i) \mid i \in I)$.

Consider that $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \le j))$ is a direct system of multialgebras from $\mathbf{Malg}'(1)$ (or $\mathbf{Malg}''(1)$). The direct limit $(H_\infty, f)$ of $\mathcal{H}$ in $\mathbf{Malg}(1)$ is a monounary multialgebra which determines the direct limit in $\mathbf{Malg}(2)$ of the direct system of hypergroupoids $((H_i)_{f_i} \mid i \in I)$. Denote by $(H_\infty, \circ)$ the resulting hypergroupoid and we can write $(H_\infty)_f = (H_\infty, \circ)$. Since each $(H_i)_{f_i}$ is a semihypergroup (or a hypergroup, respectively), $(H_\infty, \circ)$ is also semihypergroup (or a hypergroup, respectively) (see [7, Theorems 3 and 4]) and since each $(H_i)_{f_i}$ satisfies condition (3) the semihypergroup (the hypergroup, respectively) $(H_\infty, \circ)$ satisfies (3). Thus $(H_\infty, f) = G'(H_\infty, \circ)$ is the monounary multialgebra on which determines the semihypergroup (the hypergroup, respectively) $(H_\infty, \circ)$ and we have proved the following results:

**Corollary 8.** *The subcategory* $\mathbf{Malg}'(1)$ *of* $\mathbf{Malg}(1)$ *is closed under direct limits of direct systems. Moreover, if* $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *is a direct system of multialgebras from* $\mathbf{Malg}'(1)$ *and* $F'(\mathcal{H})$ *is the direct system of semihypergroups* $((F'(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *then* $F'(\varinjlim \mathcal{H})$ *is the direct limit of* $F'(\mathcal{H})$ *in* $\mathbf{Malg}(2)$.

**Corollary 9.** *The subcategory* $\mathbf{Malg}''(1)$ *of* $\mathbf{Malg}(1)$ *is closed under direct limits of direct systems. Moreover, if* $\mathcal{H} = (((H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *is a direct system of multialgebras from* $\mathbf{Malg}''(1)$ *and* $F''(\mathcal{H})$ *is the direct system of semihypergroups* $((F''(H_i, f_i) \mid i \in I), (\varphi_{ij} \mid i, j \in I, i \le j))$ *then* $F''(\varinjlim \mathcal{H})$ *is the direct limit of* $F''(\mathcal{H})$ *in* $\mathbf{Malg}(2)$.

Let $(I, \le)$ be a directed partially ordered set and let

$$\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \le j))$$

be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J, \le)$ is also a directed partially ordered set. Denote by $\mathcal{A}_J$ the direct system consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J, \le)$ and the homomorphisms are $(\varphi_{ij} \mid i, j \in J,\ i \le j)$.

**Proposition 7.** [5, Proposition 22] *Let $\mathcal{A}$ be a direct system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. Then the multialgebras $\varinjlim \mathcal{A}$ and $\varinjlim \mathcal{A}_J$ are isomorphic.*

Using the above Proposition, we get the following properties:

**Corollary 10.** *Let $(I, \leq)$ be a directed partially ordered set and $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. If $((H_i, f_i) \mid i \in I)$ is a direct system of monounary multialgebras and for any $i \in J$, $(H_i, f_i)$ satisfies the conditions ii), iii) from Proposition 4 then the direct limit multialgebra $\varinjlim_{i\in I}(H_i, f_i)$ satisfies the conditions ii), iii) from Proposition 4. The hypergroupoid determined by the monounary multialgebra $\varinjlim_{i\in I}(H_i, f_i)$ is a semihypergroup which is the direct limit of the semihypergroups $((H_i)_{f_i} \mid i \in J)$ in* $\mathbf{Malg}(2)$.

**Corollary 11.** *Let $(I, \leq)$ be a directed partially ordered set and $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. If $((H_i, f_i) \mid i \in I)$ is a direct system of monounary multialgebras and for any $i \in J$, $(H_i, f_i)$ satisfies the conditions i), ii), iii) from Proposition 4 then the direct limit multialgebra $\varinjlim_{i\in I}(H_i, f_i)$ satisfies the conditions i), ii), iii) from Proposition 4. The hypergroupoid determined by the monounary multialgebra $\varinjlim_{i\in I}(H_i, f_i)$ is a hypergroup which is the direct limit of the hypergroups $((H_i)_{f_i} \mid i \in J)$ in* $\mathbf{Malg}(2)$.

*Remark* 7. In [1, Theorem 2.8], Corsini considers a direct system of relational systems with one binary relation whose (partially ordered) carrier has a cofinal subset for which all the binary relations determines hypergroups. From the previous corollary follows not only the fact that the direct limit of the given direct system of relational system determines a hypergroup (fact proved by Corsini), but it also results that this hypergroup is the hypergroup obtained as the direct limit of the resulting direct system of hypergroups.

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