Identities in multialgebra theory

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Abstract

Adapting the notion of identity from universal algebras to multialgebras allows to provide a common language and a common approach to many important topics from multialgebra theory. In this survey article we will present some of our results on identities and multialgebras and a lot of remarks and examples which will support (in our opinion) this statement.

1. Introduction

One of the most important results concerning multialgebras is G. Grätzer's characterization theorem, which proves in [4] that the study of multialgebras is a natural extension of the theory of universal algebras. One of the problems suggested by Grätzer in his paper is the following: *What are the factor multialgebras of a group, abelian group, lattice, ring and so on? Characterize these with a suitable axiom system.* This problem as well as many of the definitions of the hyperstructures from [2] and T. Vougiouklis's works (for instance, [21]) pointed out the necessity to study the identities for multialgebras. We can talk about

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(at least) two types of identities for multialgebras: (strong) identities and weak identities. In this survey article we will refer to the following topics concerning the identities in the hyperstructure theory:

1. *How weak and/or strong identities determine some particular hyperstructures?*
2. *How identities acts with respect to some constructions of multialgebras?*

1. Some multialgebras such as semihypergroups, H_v-semigroups, hypergroups, H_v-groups, canonical hypergroups, different types of hyperrings and H_v-rings, H_v-modules are determined by weak and/or strong identities.

2. The construction of factor multialgebras is one of the first and, maybe, the most important construction in multialgebra theory. The study of the term functions of the factor multialgebra of a universal algebra leads us to an answer to the first part of Grätzer's problem, since the identities of a universal algebra usually become weak identities on the factor multialgebra. An important role in hyperstructure theory is played by those (ideal) equivalence relations for which the factor multialgebras are universal algebras. A series of important works on these equivalences of hypergroupoids, semihypergroups, hypergroups, hyperrings have been published after 1990 and converge towards the study of the fundamental relation. We gave a characterization of these equivalence relations of a multialgebra, we determined the fundamental relation of a (general) multialgebra, and we proved that the (weak) identities of a multialgebra become identities of the universal algebra obtained by factorization modulo this kind of equivalence relations. We also determined the smallest equivalence for which the factor multialgebra is a universal algebra for which a given identity is verified and we proved that the factorization of a universal algebra modulo an equivalence relation, which gave rise to multialgebras, can be seen as an intermediate step of such a factorization. As in the case of universal algebras, for a family of multialgebras which satisfy a certain (weak) identity the direct product satisfies the same identity. A similar result holds for direct limits of direct systems of multialgebras.

The main purpose of this survey article is to point out some general results which already exist in multialgebra theory and which do not need to be "rediscovered" on particular hyperstructures, but may be adapted and improved. This is why a big part of the presentation consists in remarks and examples. The proofs and other details concerning our results which appear in this paper can be found in [10], [11], [12], [13], [14], and [15]

2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence of nonnegative integers ($o(\tau)$ is an ordinal) and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation. Denote by

$$\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$$

the algebra of the n-ary terms (of type τ).

If A is a set, we denote by $P(A)$ the set of the subsets of A and by $P^*(A)$ the set of the nonempty subsets of A.

Definition 1. A *multialgebra* $\mathfrak{A}$ *of type* τ consists in a set A and a family of multioperations $(f_\gamma)_{\gamma<o(\tau)}$, where

$$f_\gamma : A^{n_\gamma} \to P^*(A)$$

is the n_γ-ary multioperation which corresponds to the symbol $\mathbf{f}_\gamma$.

It is easy to observe that if the multialgebra $\mathfrak{A}$ has no nullary multioperations, we can allow the set A to be empty, and that universal algebras are particular multialgebras.

Remark 1. A multialgebra $\mathfrak{A} = (f_\gamma)_{\gamma<o(\tau)}$ can be seen as in [20] as a relational system $(A, (r_\gamma)_{\gamma<o(\tau)})$, where r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \dots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \;\Leftrightarrow\; a_{n_\gamma} \in f_\gamma(a_0, \dots, a_{n_\gamma-1}).$$

If we allow the multioperations to be defined into $P(A)$ instead of $P^*(A)$ then any relational system can be modelled as a multialgebra using the above equivalence. It must be mentioned that in the literature there are many important papers dealing with this kind of multialgebras. From our point of view, the relational systems modelled as multialgebras are partial multialgebras, they don't satisfy the characterization theorem from [4], and most of the results we will present here are not valid for them in this form.

Remark 2. As in [16], a multialgebra $\mathfrak{A}$ determines a universal algebra $\mathfrak{P}^*(A)$ on $P^*(A)$ defining for any $\gamma < o(\tau)$, and any $A_0, \dots, A_{n_\gamma-1} \in P^*(A)$,

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid \; a_i \in A_i,\; i \in \{0, \dots, n_\gamma - 1\}\}.$$

We call $\mathfrak{P}^*(\mathfrak{A})$ *the (universal) algebra of the nonempty subsets of* $\mathfrak{A}$.

Definition 2. Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. We say that the n-ary (*strong*) *identity*

$$\mathbf{q} = \mathbf{r}$$

is satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) = r(a_0, \dots, a_{n-1}),\; \forall a_0, \dots, a_{n-1} \in A.$$

We say that the *weak identity*

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) \cap r(a_0, \dots, a_{n-1}) \neq \emptyset,\; \forall a_0, \dots, a_{n-1} \in A$$

(q and r denote the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$).

Since $\mathfrak{P}^*(\mathfrak{A})$ is a universal algebra, one can construct the algebra of the n-ary polynomial functions of $\mathfrak{P}^*(\mathfrak{A})$ (see, for instance, [1]). We denote it by $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ and we denote by $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ its subalgebra generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \dots, n-1\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are defined by

$$c^n_a(A_0, \dots, A_{n-1}) = \{a\} \text{ and } e^n_i(A_0, \dots, A_{n-1}) = A_i.$$

Of course, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$ is the subalgebra of $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ generated by $\{e^n_i \mid i \in \{0, \dots, n-1\}\}$.

3. A different approach of hypergroups, and not only ...

It is known that a *semihypergroup* $(H, \circ)$ is a multialgebra with one binary multioperation satisfying the identity

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 = \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2), \tag{1}$$

and that an H_v-*semihypergroup* can be defined in a similar way replacing (1) with

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 \cap \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \neq \emptyset. \tag{1'}$$

The following question seems to arise naturally in the context of our discussion: *Can we characterize hypergroups (H_v-groups) using (weak and/or strong) identities?* If the answer were yes, then we can characterize some particular hypergroups, some hyperrings, H_v-rings, hypermodules ... using identities.

Definition 3. Let $H \neq \emptyset$. A *hypergroup* $(H, \circ)$ is a semihypergroup which satisfies the reproductibility condition:

$$a \circ H = H \circ a = H, \ \forall a \in H.$$

From Definition 3 it follows that the maps $/, \backslash : H \times H \to P^*(H)$ defined by

$$b/a = \{x \in H \mid b \in x \circ a\}, \ a\backslash b = \{x \in H \mid b \in a \circ x\}, \tag{2}$$

are two binary multioperations on H. So, a hypergroup $(H, \circ)$ can be seen as a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations, with $H \neq \emptyset$, which satisfy (1) and (2).

Remark 3. In [12] we analyzed the reproductibility condition from the point of view of hyperoperations from (2), and we deduced that we can see a hypergroup as a nonempty multialgebra $(H, \circ, /, \backslash)$ of type $(2, 2, 2)$ which satisfy (1) and

$$\mathbf{x}_1 \cap (\mathbf{x}_1/\mathbf{x}_0) \circ \mathbf{x}_0 \neq \emptyset, \ \mathbf{x}_1 \cap \mathbf{x}_0 \circ (\mathbf{x}_0\backslash\mathbf{x}_1) \neq \emptyset. \tag{3}$$

It is not hard to observe that this multialgebra also satisfies the identities

$$\mathbf{x}_1 \cap (\mathbf{x}_1 \circ \mathbf{x}_0)/\mathbf{x}_0 \neq \emptyset, \ \mathbf{x}_1 \cap \mathbf{x}_0\backslash(\mathbf{x}_0 \circ \mathbf{x}_1) \neq \emptyset. \tag{4}$$

If in a semihypergroup the identities (3) are satisfied then the reproductibility condition holds, thus we obtain the following characterization of hypergroups:

Proposition 1. *A nonempty semihypergroup* $(H, \circ)$ *is a hypergroup if and only if there exist two binary multioperations* $/, \backslash$ *on* H *such that the multialgebra* $(H, \circ, /, \backslash)$ *satisfies the weak identities* (3).

Corollary 1. *A nonempty hypergroupoid* $(H, \circ)$ *is an* H_v*-group if and only if there exist two binary multioperations* $/, \backslash$ *on* H *such that the multialgebra* $(H, \circ, /, \backslash)$ *satisfies the weak identities* $(1')$ *and* (3).

We will see that in some cases the above characterization of hypergroups is very useful. Yet, in some cases, some of them presented herein after, it is not good enough because we cannot be sure that the existence of $/$ and $\backslash$ satisfying (3) on a semihypergroup $(H, \circ)$ does not mean necessarily that $/$ and $\backslash$ are the multioperations defined by (2).

Example 1. If $(H, \circ)$ is a group, $|H| \geq 2$, and we consider the multioperations $/, \backslash : H \times H \to P^*(H)$, defined by

$$a/b = a \backslash b = H, \ \forall a, b \in H.$$

$(H, \circ)$ is hypergroup, the identities (3) and (4) hold on $(H, \circ, /, \backslash)$, but the sets

$$\{x \in H \mid a \in x \circ b\} = \{x \in H \mid a = x \circ b\},$$
$$\{x \in H \mid a \in b \circ x\} = \{x \in H \mid a = b \circ x\}$$

are singletons for any $a, b \in H$, so, they cannot be H.

An important class of hypergroups whose definition is very close to the usual abelian group definition is the class of canonical (or reversible abelian) hypergroups. The definition we are dealing with is the one used in [18].

Definition 4. A set H with a binary multioperation $+ : H \times H \to P^*(H)$ is a *canonical hypergroup* if it verifies the following conditions:

(i) the multioperation $+$ is associative and commutative;

(ii) there exists an element $0 \in H$ such that $0 + a = a$ for all $a \in H$;

(iii) for each $a \in H$, there exists an element $-a \in H$ such that the following reversibility condition holds: for any $b, c \in H$, if $c \in a + b$ then $b \in (-a) + c$.

Clearly, the set H is not empty and $(H, +)$ is a hypergroup having the following properties:

Lemma 1. *[18, Lemma 1.1] The element* 0 *is the unique element which have the required property, and for each* $a \in H$*,* a^{-1} *is the unique element in* H *satisfying the above requirements.*

The element 0 is called *the identity element* of the canonical hypergroup $(H, +)$ and for each $a \in H$, the element a^{-1}, called *the inverse of* a, is the only element in H such that $0 \in a + (-a)$.

Corollary 2. *Canonical hypergroups can be seen as multialgebras* $(H, +, /, \backslash, 0, -)$ *where* $+, /, \backslash$ *are binary multioperations,* 0 *is a nullary operation, and* $-$ *is a unary operation, satisfying the identities* (1)*,* (3) *and*

$$\mathbf{x}_0 + \mathbf{x}_1 = \mathbf{x}_1 + \mathbf{x}_0,\ \mathbf{x}_0 + 0 = \mathbf{x}_0,\ \mathbf{x}_0/\mathbf{x}_1 = -(\mathbf{x}_1/\mathbf{x}_0).$$

Remark 4. The identity $\mathbf{x}_0/\mathbf{x}_1 = -(\mathbf{x}_1/\mathbf{x}_0)$ is the translation of the reversibility condition from Definition 4 in the language of (2).

Remark 5. Let $(H, +, /, \backslash, 0, -)$ be a multialgebra with the binary multioperations $+, /, \backslash$, the nullary operation 0, and the unary operation $-$, satisfying the above identities. If we want to verify if the multialgebra $(H, +)$ is a canonical hypergroup it seems necessary to prove first that $/$ and $\backslash$ are obtained from $+$ using the equalities (2). Otherwise, we are not sure that the satisfiability of the identity $\mathbf{x}_0/\mathbf{x}_1 = -(\mathbf{x}_1/\mathbf{x}_0)$ ensure the condition (iii) from Definition 4.

Now we are prepared to approach in a similar way other hypergroup based hyperstructures, like those which generalize rings and modules. Here we will talk only about some generalizations of rings which can be found in [9], [7] and [21], but a similar approach works for different types of hypermodules.

Definition 5. A set A with a binary multioperation $+$ and a binary operation $\cdot$ is a *hyperringoid* if:

(i) $(A, +)$ is a hypergroup;

(ii) $(A, \cdot)$ is a semigroup;

(iii) the operation $\cdot$ is distributive with respect to $+$, i.e.

$$a \cdot (b + c) = a \cdot b + a \cdot c,\ (b + c) \cdot a = b \cdot a + c \cdot a,\ \forall a, b, c \in A.$$

Remark 6. A hyperringoid can be seen as a multialgebra $(A, +, /, \backslash, \cdot)$ with three binary multioperations $+, /, \backslash$ and a binary operation $\cdot$ satisfying the following identities:

$$(\mathbf{x}_0 + \mathbf{x}_1) + \mathbf{x}_2 = \mathbf{x}_0 + (\mathbf{x}_1 + \mathbf{x}_2),\ \mathbf{x}_1 \cap (\mathbf{x}_1/\mathbf{x}_0) + \mathbf{x}_0 \neq \emptyset,\ \mathbf{x}_1 \cap \mathbf{x}_0 + (\mathbf{x}_0 \backslash \mathbf{x}_1) \neq \emptyset,$$

$$(\mathbf{x}_0 \cdot \mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_0 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_2),$$

$$\mathbf{x}_0 \cdot (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_0 \cdot \mathbf{x}_1 + \mathbf{x}_0 \cdot \mathbf{x}_2,\ (\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{x}_0 = \mathbf{x}_1 \cdot \mathbf{x}_0 + \mathbf{x}_2 \cdot \mathbf{x}_0.$$

It is also clear that if a multialgebra $(A, +, /, \backslash, \cdot)$ with three binary multioperations $+, /, \backslash$ and a binary operation $\cdot$ satisfies the above identities then $(A, +, \cdot)$ is a hyperringoid, but is not necessary that $/, \backslash$ be obtained from $+$ using (2).

Definition 6. A set A with a binary multioperation $+$ and a binary operation $\cdot$ is a *(Krasner) hyperring* if the following conditions hold:

i) $(A,+)$ is a canonical hypergroup having 0 as identity element;

ii) $(A,\cdot)$ is a semigroup;

iii) $0\cdot a = a\cdot 0 = 0$, for all $a\in A$;

iv) the operation $\cdot$ is distributive with respect to the multioperation $+$.

Remark 7. Krasner's hyperrings can be seen as multialgebras $(A,+,/,\backslash,0,-,\cdot)$ with three binary multioperations $+,/,\backslash$, a nullary operation 0, a unary operation $-$, and a binary operation $\cdot$ satisfying the identities:

$$(\mathbf{x}_0+\mathbf{x}_1)+\mathbf{x}_2=\mathbf{x}_0+(\mathbf{x}_1+\mathbf{x}_2),\ \mathbf{x}_1\cap(\mathbf{x}_1/\mathbf{x}_0)+\mathbf{x}_0\neq\emptyset,\ \mathbf{x}_1\cap\mathbf{x}_0+(\mathbf{x}_0\backslash\mathbf{x}_1)\neq\emptyset,$$

$$\mathbf{x}_0+\mathbf{x}_1=\mathbf{x}_1+\mathbf{x}_0,\ \mathbf{x}_0+0=\mathbf{x}_0,\ \mathbf{x}_0/\mathbf{x}_1=-(\mathbf{x}_1/\mathbf{x}_0),$$

$$(\mathbf{x}_0\cdot\mathbf{x}_1)\cdot\mathbf{x}_2=\mathbf{x}_0\cdot(\mathbf{x}_1\cdot\mathbf{x}_2),\ \mathbf{x}_0\cdot 0=0,\ 0\cdot\mathbf{x}_0=0$$

$$\mathbf{x}_0\cdot(\mathbf{x}_1+\mathbf{x}_2)=\mathbf{x}_0\cdot\mathbf{x}_1+\mathbf{x}_0\cdot\mathbf{x}_2,\ (\mathbf{x}_1+\mathbf{x}_2)\cdot\mathbf{x}_0=\mathbf{x}_1\cdot\mathbf{x}_0+\mathbf{x}_2\cdot\mathbf{x}_0.$$

As for a multialgebra $(A,+,/,\backslash,0,-,\cdot)$ with three binary multioperations $+,/,\backslash$, a nullary operation 0, a unary operation $-$, and a binary operation $\cdot$ satisfying the above identities, to verify if $(A,+,\cdot)$ is a (Krasner) hyperring, it seems necessary to show that $/,\backslash$ are obtained from $+$ using (2).

Definition 7. A set A with two binary multioperations $+$ and $\cdot$ is called H_v*-ring* if $(A,+)$ is an H_v-group, $(A,\cdot)$ is an H_v-semigroup, and

$$a(b+c)\cap(ab+ac)\neq\emptyset \text{ and } (b+c)a\cap(ba+ca)\neq\emptyset$$

for any $a,b,c\in A$.

Remark 8. An H_v-ring can be seen as a multialgebra $(A,+,/,\backslash,\cdot)$ with four binary multioperations $+,/,\backslash,\cdot$ satisfying the following weak identities:

$$(\mathbf{x}_0+\mathbf{x}_1)+\mathbf{x}_2\cap\mathbf{x}_0+(\mathbf{x}_1+\mathbf{x}_2)\neq\emptyset,\ \mathbf{x}_1\cap(\mathbf{x}_1/\mathbf{x}_0)+\mathbf{x}_0\neq\emptyset,\ \mathbf{x}_1\cap\mathbf{x}_0+(\mathbf{x}_0\backslash\mathbf{x}_1)\neq\emptyset,$$

$$(\mathbf{x}_0\cdot\mathbf{x}_1)\cdot\mathbf{x}_2\cap\mathbf{x}_0\cdot(\mathbf{x}_1\cdot\mathbf{x}_2)\neq\emptyset,$$

$$\mathbf{x}_0\cdot(\mathbf{x}_1+\mathbf{x}_2)\cap\mathbf{x}_0\cdot\mathbf{x}_1+\mathbf{x}_0\cdot\mathbf{x}_2\neq\emptyset,\ (\mathbf{x}_1+\mathbf{x}_2)\cdot\mathbf{x}_0\cap\mathbf{x}_1\cdot\mathbf{x}_0+\mathbf{x}_2\cdot\mathbf{x}_0\neq\emptyset.$$

Conversely, if a multialgebra $(A,+,/,\backslash,\cdot)$ satisfies the above identities, then $(A,+,\cdot)$ is an H_v-ring but, again, is not necessary that $/,\backslash$ be obtained from $+$ using (2).

Remark 9. It is clear that in a semihypergroup $(A,+)$ the construction of the term functions from $P^{(n)}(A,+)$ $(n\in\mathbb{N}^*)$ is easy. The distributivity makes this construction quite easy also in the case of hyperringoids and hyperrings. The things are from afar more complicated if $+$ is only weak associative and situation is even more complicated if we are dealing with H_v-rings.

4. Identities and factor multialgebras

For an equivalence relation ρ on a set A, we denote by $\rho\langle x\rangle$ the class of x modulo ρ, and $A/\rho = \{\rho\langle x\rangle \mid x \in A\}$.

Definition 8. Let $\mathfrak{A}$ be a multialgebra of type τ and let ρ be an equivalence relation on A. Taking for each $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1}),\ a_i\rho b_i,\ i = 0, \ldots, n_\gamma - 1\},$$

one obtains a multialgebra $\mathfrak{A}/\rho$ on A/ρ called *the factor multialgebra of* $\mathfrak{A}$ *determined by* ρ.

Remark 10. Clearly, if $\mathfrak{A}$ is a universal algebra then the definition of the multioperations in the factor multialgebra can be rewritten as follows:

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b = f_\gamma(b_0, \ldots, b_{n_\gamma-1}),\ a_i\rho b_i,\ i = 0, \ldots, n_\gamma - 1\}.$$

This is one of the most important construction of multialgebras, not only because in 1934 the first hypergroup mentioned in the literature appeared this way, but also because in 1962 it was proved that any multialgebra is obtained from such a factorization of a universal algebra.

4.1. Factor multialgebras of universal algebras

Maybe the most important result in multialgebra theory is the following:

Theorem 1. *[4] Any multialgebra which has no nullary multioperations or for which the nullary multioperations are operations is (isomorphic to) a factor of a universal algebra modulo an equivalence relation.*

As a matter of fact, the form of the above theorem from [4] is slightly different, but it was stated in this version according to some important remarks from [6]. In [4], G. Grätzer also stated the following problem: *What are the factor multialgebras of a group, abelian group, lattice, ring and so on? Characterize these with a suitable axiom system.*

To give an answer to the question in the first part of the problem we restated it as follows: *What happens with an identity of an universal algebra* $\mathfrak{A}$ *when we factorize it modulo an equivalence relation* ρ*?*

It is easy to prove that if $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ and the identity $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{A}$ then for any $a_0, \ldots, a_{n-1} \in A$ the class of

$$q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1})$$

modulo ρ is in $q(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle) \cap r(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle)$ (see [15, Remark 8]). Thus, the answer we found was that *the identities of a multialgebra become, in general, weak identities in the factor multialgebra.*

Proposition 2. *Let $\mathfrak{A}$ be a universal algebra and let ρ be an equivalence relation on A. If $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ and $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{A}$ then the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on $\mathfrak{A}/\rho$.*

Let $(G, \cdot)$ be a group. The sets

$$(2') \qquad b/a = \{x \in G \mid b = x \cdot a\},\ a\backslash b = \{x \in G \mid b = a \cdot x\}$$

are one-element sets, and the group G group can be seen as a nonempty universal algebra $(G, \cdot, /, \backslash)$ satisfying the identities

$$(\mathbf{x}_0 \cdot \mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_0 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_2),\ \mathbf{x}_1 = \mathbf{x}_0 \cdot (\mathbf{x}_0 \backslash \mathbf{x}_1),\ \mathbf{x}_1 = (\mathbf{x}_1/\mathbf{x}_0) \cdot \mathbf{x}_0,$$

$$\mathbf{x}_1 = \mathbf{x}_0 \backslash (\mathbf{x}_0 \cdot \mathbf{x}_1),\ \mathbf{x}_1 = (\mathbf{x}_1 \cdot \mathbf{x}_0)/\mathbf{x}$$

(see [17, p. 215, Exercices 8, 9]). So, if ρ is an equivalence relation on G, according to Proposition 2, the factor multialgebra $(G/\rho, \cdot, /, \backslash)$ satisfies the above identities in a weak manner. Thus, using Corollary 1, we deduce the following.

Corollary 3. *The factor of a group is an H_v-group.*

Moreover, using the above notations, we have:

Proposition 3. *In the factor multialgebra $(G/\rho, \cdot, /, \backslash)$, the multioperations $/, \backslash$ and $\cdot$ verifies the equalities* (2).

Proof. Let us denote by $//$ and $\backslash\backslash$ the multioperations defined on G/ρ by the equalities (2), i.e.

$$\rho\langle b\rangle // \rho\langle a\rangle = \{\rho\langle x\rangle \in G/\rho \mid \rho\langle b\rangle \in \rho\langle x\rangle \cdot \rho\langle a\rangle\},$$

$$\rho\langle a\rangle \backslash\backslash \rho\langle b\rangle = \{\rho\langle x\rangle \in G/\rho \mid \rho\langle b\rangle \in \rho\langle a\rangle \cdot \rho\langle x\rangle\}.$$

According to Remark 10, the multioperations $/$ and $\backslash$ are defined as follows:

$$\rho\langle b\rangle / \rho\langle a\rangle = \{\rho\langle x\rangle \in G/\rho \mid x = b'/a',\ a\rho a',\ b\rho b'\},$$

$$\rho\langle a\rangle \backslash \rho\langle b\rangle = \{\rho\langle x\rangle \in G/\rho \mid x = a'\backslash b',\ a\rho a',\ b\rho b'\}.$$

If $\rho\langle x\rangle \in \rho\langle b\rangle // \rho\langle a\rangle$ then $\rho\langle b\rangle \in \rho\langle x\rangle \cdot \rho\langle a\rangle$. Since

$$\rho\langle x\rangle \cdot \rho\langle a\rangle = \{\rho\langle b'\rangle \mid b' = x' \cdot a',\ x\rho x',\ a\rho a'\},$$

there exist $a', b', x' \in G$ such that $b' = x' \cdot a'$, $x\rho x'$, $a\rho a'$ and $b\rho b'$. But then $x' = b'/a'$, which implies $\rho\langle x\rangle = \rho\langle x'\rangle \in \rho\langle b\rangle/\rho\langle a\rangle$. Conversely, if $\rho\langle x\rangle \in \rho\langle b\rangle/\rho\langle a\rangle$ then there exist $a', b' \in G$ such that $a\rho a'$, $b\rho b'$ and $x = b'/a'$. It results that $b' = x \cdot a'$, hence $\rho\langle b\rangle = \rho\langle b'\rangle \in \rho\langle x\rangle \cdot \rho\langle a\rangle$, which implies $\rho\langle x\rangle \in \rho\langle b\rangle // \rho\langle a\rangle$.

So, we proved that $// = /$. In the same way it results that $\backslash\backslash = \backslash$. □

Remark 11. If we consider the usual definition of group, from Proposition 2 we deduce that the factor H_v-group of a group also "weakly" preserves the existence of an identity element and the existence of an inverse for each element (see [15, p. 127]).

It may be interesting to investigate if and when an identity is strongly preserved in the factor multialgebra of a universal algebra. For instance, in general, the (strong) associativity of a binary operation is not satisfied in the factor multialgebra (see [15, Example 1]). Yet, the identities which characterize the commutativity of an operation of a universal algebra hold strongly on the factor multialgebra (see [15, Example 2]). Hence the "strongly" preservation of some identity in the factor multialgebra depends on the identity. But the "strongly" preservation of some identity in the factor multialgebra depends also on the equivalence relation we are factorizing with. To show this, we consider the following example.

Example 2. If $(G, \cdot)$ is a finite group and ρ is the equivalence relation

$$x\rho y \Leftrightarrow \exists g \in G : y = g^{-1}xg$$

(i.e. ρ is the conjugacy relation on G) then $\overline{G} = G/\rho$ is the set of the conjugacy classes of G, and the definition of $\cdot$ in G/ρ can be also written as follows

$$C_i \cdot C_j = \{C_k \in \overline{G} \mid \exists g_i \in C_i, \exists g_j \in C_j : \; g_i \cdot g_j \in C_k\},$$

for any $C_i, C_j \in \overline{G}$. The hypergroupoid $(\overline{G}, \cdot)$ is a canonical hypergroup having the identity element $\rho\langle 1\rangle = \{1\}$, and for each conjugacy class C from G, the inverse element is the class C^{-1} which consists in the inverses of the elements of C (see [18]). However, let us consider $/$ and $\backslash$ as in $(2')$, 1 the nullary operation which points out the identity element, $^{-1}$ the unary operation which associates to each element its inverse. Let us take the universal algebra $(G, \cdot, /, \backslash, 1, ^{-1})$ and the factor multialgebra $(G/\rho, \cdot, /, \backslash, \rho\langle 1\rangle, ^{-1})$. Of course, $\rho\langle 1\rangle$ and $^{-1}$ are operation on G/ρ, too. According to Proposition 3, $/$, $\backslash$ and $\cdot$ satisfy (2), thus the following identities are satisfied (in a strong manner) both on $(G, \cdot, /, \backslash, 1, ^{-1})$ and $(G/\rho, \cdot, /, \backslash, \rho\langle 1\rangle, ^{-1})$:

$$(\mathbf{x}_0 \cdot \mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_0 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_2), \;\; \mathbf{x}_0 \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{x}_0 = \mathbf{x}_0, \;\; \mathbf{x}_0/\mathbf{x}_1 = -(\mathbf{x}_1/\mathbf{x}_0).$$

Another consequence of Proposition 2 is:

Corollary 4. *The factor of a ring $(R, +, \cdot)$ is an H_v-ring.*

Of course, there are more identities than those in the definition of an H_v-ring which are "weakly" preserved in the factor H_v-ring of a ring, and even "strongly" preserved if we are thinking to the commutativity of $+$.

Remark 12. The factor multialgebra of a lattice is not necessarily a hyperlattice, since the absorption is required in a strong manner in the definition of a hyperlattice, and, in general, it is preserved only in a weak manner by the factorization of a multialgebra (see [15, Example 3]).

4.2. Factor multialgebras which are universal algebras

Let ρ be an equivalence relation on a set A. Let us remember that ρ induces a relation $\overline{\overline{\rho}}$ on $P^*(A)$ as follows: for $X, Y \in P^*(A)$,

$$X \overline{\overline{\rho}} Y \;\Leftrightarrow\; x\rho y,\ \forall x \in X,\ \forall y \in Y\ (\Leftrightarrow\ X \times Y \subseteq \rho).$$

Proposition 4. *[15, Proposition 4.1] Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra and let ρ be an equivalence relation on A. The following conditions are equivalent:*

a) $\mathfrak{A}/\rho$ is a universal algebra;

b) If $\gamma < o(\tau)$, $a, b, x_i \in A$, $i \in \{0, \ldots, n_\gamma - 1\}$, with $a\rho b$, then

$$f_\gamma(x_0, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_\gamma-1}) \overline{\overline{\rho}} f_\gamma(x_0, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_\gamma-1})$$

for all $i \in \{0, \ldots, n_\gamma - 1\}$;

c) If $\gamma < o(\tau)$, $x_i, y_i \in A$ and $x_i \rho y_i$ for any $i \in \{0, \ldots, n_\gamma - 1\}$, then

$$f_\gamma(x_0, \ldots, x_{n_\gamma-1}) \overline{\overline{\rho}} f_\gamma(y_0, \ldots, y_{n_\gamma-1});$$

d) If $n \in \mathbb{N}$, $\mathbf{p} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$, $x_i, y_i \in A$ and $x_i \rho y_i$ for all $i \in \{0, \ldots, n-1\}$, then

$$p(x_0, \ldots, x_{n-1}) \overline{\overline{\rho}} p(y_0, \ldots, y_{n-1}).$$

Example 3. An equivalence relation ρ on a hypergroupoid $(H, \circ)$ is called *strongly regular* if for any $a, b, x \in H$ with $a\rho b$ we have $a \circ x \overline{\overline{\rho}} b \circ x$ and $x \circ a \overline{\overline{\rho}} x \circ b$ (see [2]). It is clear that the strongly regular equivalences of a hypergroupoid are those relations ρ for which $(H/\rho, \cdot)$ is a groupoid.

Notation 1. We denote by $E_{ua}(\mathfrak{A})$ the set of the relations characterized in Proposition 4.

Remark 13. If $\rho \in E_{ua}(\mathfrak{A})$ then in the factor multialgebra $\mathfrak{A}/\rho$, which is a universal algebra, we have

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma-1}\rangle) = \rho(f_\gamma(a_0, \ldots, a_{n_\gamma-1})),$$

the canonical projection is an ideal homomorphism, so, using Theorem 2 from [16], we deduce that for any $n \in \mathbb{N}$ and $\mathbf{p} \in P^{(n)}(\tau)$, we also have

$$p(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle) = \rho(p(a_0, \ldots, a_{n-1})). \tag{5}$$

As a matter of fact, condition d) from Proposition 4 allows us to rewrite (5) as follows

$$p(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n-1}\rangle) = \rho(p(a_0, \ldots, a_{n-1})) = \rho\langle b\rangle,\ \forall\, b \in p(a_0, \ldots, a_{n-1}). \tag{5'}$$

Lemma 2. *[15, Lemma 4.2] $E_{ua}(\mathfrak{A})$ is an algebraic closure system on $A \times A$.*

Corollary 5. *[15, Corollary 4.3] The smallest relation from $E_{ua}(\mathfrak{A})$ which contains a relation $R \subseteq A \times A$ is*

$$\alpha(R) = \bigcap \{\rho \in E_{ua}(\mathfrak{A}) \mid R \subseteq \rho\}.$$

Corollary 6. *[15, Remark 14] The relation $\alpha^* = \alpha(\emptyset) = \alpha(\delta_A)$ is the smallest element of $E_{ua}(\mathfrak{A})$ and it is called the fundamental relation of $\mathfrak{A}$.*

Remark 14. Given $\mathbf{q}, \mathbf{r} \in P^{(n)}(\tau)$ $(n \in \mathbb{N})$ and a multialgebra $\mathfrak{A}$, we can obtain a factor multialgebra of $\mathfrak{A}$ which is a universal algebra satisfying $\mathbf{q} = \mathbf{r}$ by taking the factor multialgebra determined by a relation ρ from $E_{ua}(\mathfrak{A})$ which contains

$$R_{\mathbf{qr}} = \bigcup \{q(a_0, \dots, a_{n-1}) \times r(a_0, \dots, a_{n-1}) \mid a_0, \dots, a_{n-1} \in A\}.$$

This happens because, using (5′), one obtains immediately that in the factor (multi)algebra $\mathfrak{A}/\rho$, $q(\rho\langle a_0\rangle, \dots, \rho\langle a_{n-1}\rangle)$ and $r(\rho\langle a_0\rangle, \dots, \rho\langle a_{n-1}\rangle)$ are equal elements. Using (5′), it also follows that each relation from $E_{ua}(\mathfrak{A})$ which gives a factor multialgebra satisfying the identity $\mathbf{q} = \mathbf{r}$ must contain the relation $R_{\mathbf{qr}}$.

In [12] we proved that the following Proposition.

Proposition 5. *Any identity (weak or strong) which holds on $\mathfrak{A}$ is also satisfied on its fundamental algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*$.*

Since α^* is the smallest element from $E_{ua}(\mathfrak{A})$, from the previous remark we get that if a weak (or strong) identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ (or $\mathbf{q} = \mathbf{r}$) holds in $\mathfrak{A}$ and $\rho \in E_{ua}(\mathfrak{A})$ then $R_{\mathbf{qr}} \subseteq \alpha^* \subseteq \rho$. Thus it follows immediately the following:

Corollary 7. *If an identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ (or $\mathbf{q} = \mathbf{r}$) is satisfied in $\mathfrak{A}$ and $\rho \in E_{ua}(\mathfrak{A})$ then the identity $\mathbf{q} = \mathbf{r}$ is satisfied in the universal algebra $\mathfrak{A}/\rho$.*

Applying the previous corollary to a semihypergroup $(H, \circ)$ and a relation $\rho \in E_{ua}(H, \circ)$ we obtain the first part of Theorem 31 from [2]:

Corollary 8. *The factor hypergroupoid of a semihypergroup modulo a strongly regular equivalence is a semihypergroup.*

Remark 15. In the second part of [2, Theorem 31] is also proved that if the semihypergroup $(H, \circ)$ is a hypergroup than the factor semihypergroup $(H/\rho, \circ)$ is a group. This means that considering the hypergroup $(H, \circ)$ as a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations as in Remark 3 we have

$$E_{ua}(H, \circ) = E_{ua}(H, \circ, /, \backslash).$$

Moreover, $/$ (and $\backslash$) determines in the factor (multi)algebra $(H/\rho, \circ, /, \backslash)$ the binary operation which associates to a pair $(\rho\langle a\rangle, \rho\langle b\rangle) \in H/\rho \times H/\rho$ the (unique) solution from H/ρ of the equation

$$\rho\langle b\rangle = x \circ \rho\langle a\rangle \text{ (and } \rho\langle b\rangle = \rho\langle a\rangle \circ x \text{ respectively).}$$

Indeed, if $a, b \in H$ then

$$\rho\langle b\rangle/\rho\langle a\rangle = \{\rho\langle c\rangle \mid c \in b'/a',\ a\rho a',\ b\rho b'\},$$

thus for $x \in \rho\langle a\rangle/\rho\langle b\rangle$ there exists $a', b', c \in H$ such that

$$\rho\langle a\rangle = \rho\langle a'\rangle,\ \rho\langle b\rangle = \rho\langle b'\rangle,\ x = \rho\langle c\rangle \text{ and } b' \in c \circ a'.$$

But then

$$\rho\langle a\rangle = \rho\langle a'\rangle = \rho\langle c\rangle \circ \rho\langle b'\rangle = x \circ \rho\langle b\rangle.$$

Since $(H/\rho, \circ)$ is a group, we deduce that for any $\rho\langle a\rangle, \rho\langle b\rangle \in H/\rho$ the set $\rho\langle b\rangle/\rho\langle a\rangle$ consists in the only element x which verifies the equality

$$\rho\langle b\rangle = x \circ \rho\langle a\rangle.$$

Analogously, we can prove the same property for $\backslash$.

As a matter of fact, in the above remark is already contained the proof of the fact that $(H/\rho, \circ)$ is a group, since the existence of solution for each of the equations $\rho\langle b\rangle = x\circ\rho\langle a\rangle$ and $\rho\langle b\rangle = \rho\langle a\rangle\circ x$ for any $\rho\langle a\rangle, \rho\langle b\rangle \in H/\rho$ is a necessary and sufficient condition for the nonempty semigroup H/ρ to be a group.

Corollary 9. *Let $(A, +, \cdot)$ be an H_v-ring. An equivalence relation ρ on A is in $E_{ua}(A, +, \cdot)$ if and only if it is strongly regular both on $(A, +)$ and $(A, \cdot)$. In this case, the factor multialgebra $(A/\rho, +, \cdot)$ is a nearring. If, in addition, $+$ is at least weak commutative, then $(A/\rho, +, \cdot)$ is a ring.*

Corollary 10. *Let $(A, +, \cdot)$ be a hyperringoid and ρ an equivalence relation on A. The relation ρ is in $E_{ua}(A, +, \cdot)$ if and only if it is strongly regular on $(A, +)$ and it is a congruence on $(A, \cdot)$. In this case, the factor multialgebra $(A/\rho, +, \cdot)$ is a nearring. If, in addition, $+$ is at least weak commutative, then $(A/\rho, +, \cdot)$ is a ring.*

Remark 16. [15, Remarks 13, 14] The smallest equivalence relation from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$ is the relation $\alpha^*_{\mathbf{qr}} = \alpha(R_{\mathbf{qr}})$. In particular, $\alpha^* = \alpha^*_{\mathbf{x}_0\mathbf{x}_0}$.

Theorem 2. *[10, Theorem 4] The fundamental relation α^* of a multialgebra $\mathfrak{A}$ is the transitive closure of the relation α defined by*

$$x\alpha y \ \Leftrightarrow\ \exists n \in \mathbb{N},\ \exists p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists a_0, \ldots, a_{n-1} \in A:\ x, y \in p(a_0, \ldots, a_{n-1}).$$

Remark 17. [5, Corollary 8.2] For any $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and for any $m \in \mathbb{N}$, $m \geq n$ there exists $q \in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

$$p(A_0, \ldots, A_{n-1}) = q(A_0, \ldots, A_{m-1}),$$

for every $A_0, \ldots, A_{m-1} \in P^*(A)$.

Remark 18. For any $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and for any permutation σ of the set $\{0, \ldots, n-1\}$ there exists $q \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

$$p(A_0, \ldots, A_{n-1}) = q(A_{\sigma(0)}, \ldots, A_{\sigma(n-1)}),\ \forall A_0, \ldots, A_{n-1} \in P^*(A).$$

Remark 19. The relation α remains the same if we can consider term functions from $P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ instead of polynomial functions from $P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ since for any polynomial function $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A})$ and any $a_0, \ldots, a_{n-1} \in A$, there exist $m \in \mathbb{N}$, $m \geq n$, $b_0, \ldots, b_{m-1} \in A$ and a term function $p' \in P_A^{(m)}(\mathfrak{P}^*(\mathfrak{A})$ such that

$$p(a_0, \ldots, a_{n-1}) = p'(b_0, \ldots, b_{m-1}).$$

Indeed, if $p = c_a^n$, we can consider $m = n+1$, $b_0 = a_0$, $\ldots$, $b_{n-1} = a_{n-1}$, $b_n = b_{m-1} = a$ and $p' = e_{m-1}^m$ and we have

$$c_a^n(a_0, \ldots, a_{n-1}) = a = b_{m-1} = e_{m-1}^m(b_0, \ldots, b_{m-1}).$$

If $p = e_i^n$ $(i \in \{0, \ldots, n-1\})$, the equality trivially holds if we consider

$$m = n,\ b_0 = a_0, \ldots,\ b_{n-1} = a_{n-1},\ p' = p.$$

Let us take $\gamma < o(\tau)$, the polynomial functions $p_0, \ldots, p_{n_\gamma - 1} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and

$$p = f_\gamma(p_0, \ldots, p_{n_\gamma - 1}).$$

Let $a_0, \ldots, a_{n-1} \in A$ and assume that for each $j \in \{0, \ldots, n_\gamma - 1\}$ there exist $m_j \in \mathbb{N}$, $n \leq m_j$, $b_0^j, \ldots, b_{m_j-1}^j \in A$ and a term function $p_j' \in P^{(m_j)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

$$p_j(a_0, \ldots, a_{n-1}) = p_j'(b_0^j, \ldots, b_{m_j-1}^j).$$

Consider $m = m_1 + \cdots + m_{n_\gamma - 1}$ and let $b_0, \ldots, b_{m-1}$ be the elements

$$b_0^0, \ldots, b_{m_0-1}^0, \ldots, b_0^{n_\gamma - 1}, \ldots, b_{m_{n_\gamma-1}-1}^{n_\gamma - 1}$$

respectively. According to Remarks 17 and 18, for each $j \in \{0, \ldots, n_\gamma - 1\}$ there exists a term function $p_j'' \in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}))$ such that

$$p_j'(b_0^j, \ldots, b_{m_j-1}^j) = p_j''(b_0, \ldots, b_{m-1}).$$

Hence for any $j \in \{0, \ldots, n_\gamma - 1\}$,

$$p_j(a_0, \ldots, a_{n-1}) = p_j''(b_0, \ldots, b_{m-1}).$$

It follows that

$$\begin{aligned} p(a_0, \ldots, a_{n-1}) &= f_\gamma(p_0, \ldots, p_{n_\gamma-1})(a_0, \ldots, a_{n-1}) \\ &= f_\gamma(p_0(a_0, \ldots, a_{n-1}), \ldots, p_{n_\gamma-1}(a_0, \ldots, a_{n-1})) \\ &= f_\gamma(p_0''(b_0, \ldots, b_{m-1}), \ldots, p_{n_\gamma-1}''(b_0, \ldots, b_{m-1})) \\ &= f_\gamma(p_0'', \ldots, p_{n_\gamma-1}'')(b_0, \ldots, b_{m-1}). \end{aligned}$$

So, if we take p' to be the term function $f_\gamma(p_0'', \ldots, p_{n_\gamma-1}'')$ we obtain the required equality.

Corollary 11. *The fundamental relation α^* of a multialgebra $\mathfrak{A}$ is the transitive closure of the relation α defined by*

$$x\alpha y \;\Leftrightarrow\; \exists n \in \mathbb{N},\ \exists p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists a_0,\dots,a_{n-1} \in A:\ x,y \in p(a_0,\dots,a_{n-1}).$$

Example 4. The fundamental relation of a semihypergroup $(H,\circ)$ is the transitive closure of the relation $\beta = \bigcup_{n\in\mathbb{N}^*} \beta_n$ where for any $x,y \in H$,

$$x\beta_n y \Leftrightarrow \exists a_0,\dots,a_{n-1} \in H:\ x,y \in a_0 \circ \cdots \circ a_{n-1}.$$

If the semihypergroup $(H,\circ)$ is a hypergroup, the relation β is already transitive, so $\beta^* = \beta$ (see [2]).

Remark 20. Let us see a hypergroup $(H,\circ)$ as a multialgebra $(H,\circ,/,\backslash)$ as in Remark 3. Remark 15 gives a reason why in the construction of the the fundamental relation of the multialgebra $(H,\circ,/,\backslash)$ (hence of a hypergroup) we do not have to consider the term functions obtained using the multioperations $/$ and $\backslash$, but only using the multioperation $\circ$.

Theorem 3. *[15, Theorem 4.4] The relation $\alpha^*_{\mathbf{qr}}$ is the transitive closure of the relation $\alpha_{\mathbf{qr}}$ defined as follows*

$$\begin{aligned} x\alpha_{\mathbf{qr}} y \;\Leftrightarrow\; & \exists p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists\ a_0,\dots,a_{n-1} \in A: \\ & x \in p(q(a_0,\dots,a_{n-1})),\ y \in p(r(a_0,\dots,a_{n-1}))\ \text{or} \\ & y \in p(q(a_0,\dots,a_{n-1})),\ x \in p(r(a_0,\dots,a_{n-1})). \end{aligned}$$

Example 5. [15, Example 5] In the case of (semi)hypergroups, taking $\mathbf{q} = \mathbf{x}_0\mathbf{x}_1$ and $\mathbf{r} = \mathbf{x}_1\mathbf{x}_0$ (i.e. the studied identity is the commutativity of the hyperproduct) the relation $\alpha^*_{\mathbf{qr}}$ is the relation γ^* introduced Freni in [3] to characterize the derived subhypergroup of a hypergroup.

4.3. Back to factor multialgebras of universal algebras

Let $n \in \mathbb{N}$, $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)$, let $\mathfrak{A}$ be a universal algebra and ρ an equivalence relation on A.

Notation 2. We denote by $\rho_{\mathbf{qr}}$ the smallest equivalence relation on A containing ρ and

$$\{(q(a_0,\dots,a_{n-1}), r(a_0,\dots,a_{n-1})) \mid a_0,\dots,a_{n-1} \in A\}.$$

We denote by $\theta(\rho_{\mathbf{qr}})$ the smallest congruence relation on $\mathfrak{A}$ containing $\rho_{\mathbf{qr}}$ and by $\theta(\rho)$ the smallest congruence relation on $\mathfrak{A}$ which contains ρ.

Theorem 4. *[15, Theorem 5.3] Let $n \in \mathbb{N}$ and $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If $\mathfrak{A}$ is a universal algebra and ρ is an equivalence relation on A then*

$$(\mathfrak{A}/\rho)/\alpha^*_{\mathbf{qr}} \cong \mathfrak{A}/\theta(\rho_{\mathbf{qr}}).$$

Corollary 12. *[15, Corollary 5.4] If $\mathfrak{A}$ is a universal algebra and ρ is an equivalence relation on A then*
$$\overline{\mathfrak{A}/\rho} \cong \mathfrak{A}/\theta(\rho).$$

Corollary 13. *[15, Corollaries 5.5, 5.6] Let $n \in \mathbb{N}$ and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If ρ is an equivalence relation on the universal algebra $\mathfrak{A}$ then*

$$\overline{\mathfrak{A}/\rho_{\mathbf{qr}}} \cong \mathfrak{A}/\theta(\rho_{\mathbf{qr}}) \cong (\mathfrak{A}/\rho)/\alpha^*_{\mathbf{qr}}.$$

Example 6. Let $(G, \cdot)$ be a group, H a subgroup of G, $G/H = \{xH \mid x \in G\}$ and consider the multioperation

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

The hypergroupoid $(G/H, \cdot)$ is mentioned in the literature as the first example of multialgebra, more exactly, the first example o hypergroup. Obviously, this hypergroup is a group if and only if H is a normal subgroup of G.

Let γ be the smallest strongly regular equivalence on G/H such that the factor hypergroup is a commutative group (see [3]). If G' is the derived subgroup of G then $G'H$ is the smallest normal subgroup N of G for which $H \subseteq N$ and G/N is abelian. From Theorem 4 we obtain the group isomorphism

$$(G/H)/\gamma \cong G/(G'H).$$

If $\overline{G/H}$ is the fundamental group of the hypergroup G/H and $\overline{H}$ is the smallest normal subgroup of G which contains H then, according to Corollary 12, we have

$$\overline{G/H} \cong G/\overline{H}.$$

5. Direct products and direct limits of direct systems

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ. The Cartesian product $\prod_{i\in I} A_i$ with the multioperations

$$f_\gamma((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma - 1})_{i\in I}) = \prod_{i\in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}),$$

is a multialgebra called *the direct product of the multialgebras* $(\mathfrak{A}_i \mid i \in I)$.

Lemma 3. *[11, Lemma 1] If $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$ then*
$$p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1}).$$

Proposition 6. *[11, Propositions 3, 4] The direct product of a family of multialgebras which satisfy a certain identity (weak or strong) satisfies the same identity.*

Using the remarks from Section 3 we have:

Corollary 14. *The direct product of a family of semihypergroups is a semihypergroup.*

Corollary 15. *The product of a family of H_v-semigroups is an H_v-semigroup.*

Corollary 16. *The direct product of a family of H_v-groups is an H_v-group.*

Corollary 17. *[13, Corollary 2] The direct product of a family of hypergroups is a hypergroup.*

From the above corollaries we obtain easily the following:

Corollary 18. *The direct product of a family of hyperringoids is a hyperringoid and the direct product of a family of H_v-rings is an H_v-ring.*

Remark 21. In the proof of [13, Corollary 2] we proved even more: if we see each hypergroup of a family $((H_i, \circ_i) \mid i \in I)$ of hypergroups as a multialgebra $(H_i, \circ_i, /_i, \backslash_i)$ as in Remark 3 and we consider the direct product multialgebra $\left(\prod_{i\in I} H_i, \circ, /, \backslash\right)$ then $\left(\prod_{i\in I} H_i, \circ\right)$ is a hypergroup and $/, \backslash$ are obtained from $\circ$ using (2). Thus the above corollaries also can be written for canonical hypergroups and Krasner's hyperrings.

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let A_∞ be the direct limit of the direct system of their supporting sets. Let us remember that:

- $(I, \leq)$ is a directed preordered set;
- the homomorphisms φ_{ij} are such that
$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik},\ \forall i, j, k \in I,\ i \leq j \leq k,$$
$$\varphi_{ii} = 1_{A_i},\ \forall i \in I.$$
- the set A_∞ is the factor of the disjoint union A of the sets A_i modulo the equivalence relation $\equiv$ defined as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and
$$x \equiv y \ \Leftrightarrow\ \exists k \in I,\ i \leq k,\ j \leq k :\ \varphi_{ik}(x) = \varphi_{jk}(y).$$

We define the multioperations f_γ on $A_\infty = \{\widehat{x} \mid x \in A\}$ as follows: if $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ and for any $j \in \{0, \ldots, n_\gamma - 1\}$ we consider that $x_j \in A_{i_j}$ $(i_j \in I)$ then

$$f_\gamma(\widehat{a_0}, \ldots, \widehat{a_{n_\gamma-1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I, i_0 \leq m, \ldots, i_{n_\gamma-1} \leq m,$$
$$a \in f_\gamma(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n_\gamma-1} m}(a_{n_\gamma-1}))\}.$$

We obtain a multialgebra $\mathfrak{A}_\infty = (A_\infty, (f_\gamma)_{\gamma<o(\tau)})$, called *the direct limit of the direct system* $\mathcal{A}$.

We mentioned that the construction of the direct limit of a direct system of multialgebras was inspired by the construction of the direct limit of a direct system of semihypergroups from [19] and equivalently changed to bring it closer to the same construction from universal algebra theory (see, for instance, [5]).

Lemma 4. *[14, Lema 28] If* $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$, $a_0, \ldots, a_{n-1} \in A$ *and* $i_0, \ldots, i_{n-1} \in I$ *are such that* $a_j \in A_{i_j}$ *for all* $j \in \{0, \ldots, n-1\}$ *then*

$$p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\, i_0 \leq m, \ldots,\, i_{n-1} \leq m,\\ a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}.$$

Proposition 7. *[14, Lemmas 29, 31] The direct limit of a direct system of multialgebras which satisfy a certain identity (weak or strong) satisfies the same identity.*

Using the remarks from Section 3 we obtain:

Corollary 19. *The direct limit of a direct system of* H_v*-semigroups is an* H_v*-semigroup and the direct limit of a direct system of* H_v*-groups is an* H_v*-group.*

Corollary 20. *[19, Theorem 3] The direct limit of a direct system of semihypergroups is a semihypergroup.*

Corollary 21. *The direct limit of a direct system of hypergroups is a hypergroup.*

The above result appear in [19], but not quite in this form.

Corollary 22. *The direct limit of a direct system of hyperringoids is a hyperringoid and the direct limit of a direct system of* H_v*-rings is an* H_v*-ring.*

Remark 22. In [14, Remark 35] we proved that if we see each hypergroup of the given direct system $((H_i, \circ_i) \mid i \in I)$ as a multialgebra $(H_i, \circ_i, /_i, \backslash_i)$ as in Remark 3 and we consider the direct limit multialgebra $(H_\infty, \circ, /, \backslash)$ then $(H_\infty, \circ)$ is a hypergroup and $/, \backslash$ are obtained from $\circ$ using (2). Thus the above corollaries can be rewritten for canonical hypergroups and then for Krasner's hyperrings. The result containing Krasner's hyperrings is, somehow, contained in [8].

We mention that an important property of direct limits of direct systems of multialgebras is presented in Proposition 22 from [14]. It is the tool which leads us from the above results on hypergroups and Krasner hyperrings to their versions from [19] and [8].

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