MULTIALGEBRAS AND TERM FUNCTIONS OVER THE ALGEBRA OF THEIR NONVOID SUBSETS

Cosmin Pelea, Simion Breaz

Abstract. The object of this paper are multialgebras. Starting from the idea of Pickett that the set of nonvoid subsets of a multialgebra can be organized naturally as a universal algebra, and constructing the term functions over this algebra we can deduce some results on multialgebras from some of the very known properties of the term functions of a universal algebra. Also, we will present the form of the generated submultialgebra in two theorems, which, in the particular case of the universal algebras are already known.

Given $\tau = (n_0, ..., n_\gamma, ...)_{\gamma<o(\tau)}$, where $o(\tau)$ is an ordinal, and a multialgebra $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)_{\gamma<o(\tau)}$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ being a multioperation with the arity $n_\gamma \in \mathbb{N}$, for any $\gamma < o(\tau)$, we denote by $S(\mathfrak{A})$ the set of the submultialgebras of $\mathfrak{A}$.

From the results of [5] we can deduce the following statements:

1) $S(\mathfrak{A})$ is an algebraic closure system on A.

2) $\mathfrak{S}(\mathfrak{A}) = (S(\mathfrak{A}), \subseteq)$ is an algebraic lattice.

3) If $X \subseteq A$, then $\langle X\rangle = \bigcap\{B \in S(\mathfrak{A})|\ X \subseteq B\}$ with the multioperations of $\mathfrak{A}$ form a submultialgebra of $\mathfrak{A}$ called the submultialgebra of $\mathfrak{A}$ generated by the subset X. It is easy to observe that if $\mathfrak{B}$ is a submultialgebra for $\mathfrak{A}$ then $\langle B\rangle = B$.

4) $\langle\emptyset\rangle = \emptyset$ if and only if $\mathfrak{A}$ has no nullary multioperations.

5) If $\mathfrak{A}$ has nullary multioperations and A_0 is the union of the sets which are images of these multioperations, then $\emptyset \neq \langle\emptyset\rangle = \langle A_0\rangle$ is the least submultialgebra of $\mathfrak{A}$ in the sense of inclusion.

6) S is an algebraic closure system on A if and only if it exists a structure of multialgebra $\mathfrak{A}$ on A such that $S(\mathfrak{A}) = S$.

7) An ordered set $\mathfrak{L}$ is an algebraic lattice if and only if it exists a multialgebra $\mathfrak{A}$ such that $\mathfrak{L} \simeq \mathfrak{S}(\mathfrak{A})$.

Now, we can give a way to construct the generated submultialgebra of a multialgebra:

1. Theorem. *Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)_{\gamma<o(\tau)}$ be a multialgebra of type τ and $X \subseteq A$. We consider $X_0 = X$ and for any $k \in \mathbb{N}$,*

$$X_{k+1} = X_k \cup (\bigcup\{f_\gamma(x_0, ..., x_{n_\gamma-1})|\ x_0, ..., x_{n_\gamma-1} \in X_k,\ \gamma < o(\tau)\}.$$

Then: $\langle X\rangle = \bigcup_{k\in\mathbb{N}} X_k$.

Proof. Let us consider $M = \bigcup_{k\in\mathbb{N}} X_k$ and $x_0, ..., x_{n_\gamma-1} \in M$ ($\gamma < o(\tau)$ arbitrary). From $X_0 \subseteq X_1 \subseteq ... \subseteq X_k \subseteq ...$ it follows the existence of $m \in \mathbb{N}$ such that $x_0, ..., x_{n_\gamma-1} \in$

$\in X_m$, which implies, according to the definitions of X_{m+1} that $f_\gamma(x_0,...,x_{n_\gamma-1}) \subseteq X_{m+1}$. Thus $f_\gamma(x_0,...,x_{n_\gamma-1}) \subseteq M$, and $\mathfrak{M} = (M, f_0, f_1, ..., f_\gamma, ...)$ is a submultialgebra of the multialgebra $\mathfrak{A}$. From $X = X_0 \subseteq M$, by the definition of the generated submultialgebra, it results $\langle X \rangle \subseteq \langle M \rangle = M$. To prove the inverse inclusion we will show by induction on $k \in \mathbb{N}$ that $X_k \subseteq \langle X \rangle$ for any $k \in \mathbb{N}$. Indeed, we have $X_0 = X \subseteq \langle X \rangle$. We suppose that $X_k \subseteq \langle X \rangle$. From $\langle X \rangle \in S(\mathfrak{A})$ and the definition of X_{k+1} we can deduce that $X_{k+1} \subseteq \langle X \rangle$. Therefore $M \subseteq \langle X \rangle$. The two inclusions lead us to $M = \langle X \rangle$.

Let us remember that for a given multialgebra $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)_{\gamma < o(\tau)}$, the set of the nonvoid subsets of A, $P^*(A)$, can be organized as a universal algebra with the operations:

$$f_\gamma(A_0, ..., A_{n_\gamma-1}) = \bigcup \{ f_\gamma(a_0, ..., a_{n_\gamma-1}) \mid \ a_i \in A_i, \ \forall i \in \{0, ..., n_\gamma - 1\}\},$$

for any $\gamma < \mathrm{o}(\tau)$ and $A_0, ..., A_{n_\gamma-1} \in P^*(A)$. We denote this algebra by $\mathfrak{P}^*(A)$.

In [4] Grätzer presents the algebra of the term functions of a universal algebra(in fact, the notion met in [4] is "polynomial", but in nowadays it was replaced by the notion of term function (see [1])). If we consider an algebra $\mathfrak{B} = (B, f_0, f_1, ..., f_\gamma, ...)_{\gamma < o(\tau)}$ we call n-ary term functions on $\mathfrak{B}$ ($n \in \mathbb{N}$) those and only those functions from B^n into B which can be obtained by applying (i) and (ii) from bellow for finitely many times:

(i) the functions $e_i^n : B^n \to B$, $e_i^n(x_0, ..., x_{n-1}) = x_i$, $i = 0, ..., n-1$ are n-ary term functions on $\mathfrak{B}$;

(ii) if $p_0, ..., p_{n_\gamma-1}$ are n-ary term functions on $\mathfrak{B}$ then $f_\gamma(p_0, ..., p_{n_\gamma-1}) : B^n \to B$, $(f_\gamma(p_0, ..., p_{n_\gamma-1}))(x_0, ..., x_{n-1}) = f_\gamma(p_0(x_0, ..., x_{n-1}), ..., p_{n_\gamma-1}(x_0, ..., x_{n-1}))$ is also a n-ary term function on $\mathfrak{B}$.

We can observe that (ii) organize the set of n-ary term functions on $\mathfrak{B}$ ($P^{(n)}(\mathfrak{B})$) as a universal algebra, denoted by $\mathfrak{P}^{(n)}(\mathfrak{B})$.

For any $n \in \mathbb{N}$, we can construct the algebra of n-ary term functions on $\mathfrak{P}^*(A)$, $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ (we notice that $\mathfrak{P}^{(0)}(\mathfrak{P}^*(A))$ exists only if there are nullary multioperations on A).

One of the results presented in [6] is the following:

2. Theorem. *A necessary and sufficient condition for $\mathfrak{P}^*(\mathfrak{B})$ to be a subalgebra of $\mathfrak{P}^*(\mathfrak{A})$ is that $\mathfrak{B}$ to be a submultialgebra for $\mathfrak{A}$.*

3. Corollaries. a) Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ be a multialgebra of type τ, $\mathfrak{B}$ a submultialgebra of $\mathfrak{A}$ and $p \in P^{(n)}(\mathfrak{P}^*(A))$, ($n \in \mathbb{N}$). If $B_0, ..., B_{n-1} \subseteq B$ are nonvoid parts, then $p(B_0, ..., B_{n-1}) \subseteq B$.

b) Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ be a multialgebra of type τ, $\mathfrak{B}$ a submultialgebra of $\mathfrak{A}$ and $p \in P^{(n)}(\mathfrak{P}^*(A))$, ($n \in \mathbb{N}$). If $b_0, ..., b_{n-1} \in B$ then $p(b_0, ..., b_{n-1}) \subseteq B$.

4. Theorem. *Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ be a multialgebra of type τ, $X \subseteq A$, $X \neq \emptyset$. Then $a \in \langle X \rangle$ if and only if $\exists n \in \mathbb{N}$, $\exists p \in P^{(n)}(\mathfrak{P}^*(A))$, $\exists x_0, ..., x_{n-1} \in X$ such that $a \in p(x_0, ..., x_{n-1})$.*

Proof. We denote

$$M = \bigcup\{p(x_0, ..., x_{n-1}) |\ n \in \mathbb{N},\ p \in P^{(n)}(\mathfrak{P}^*(A)),\ x_0, ..., x_{n-1} \in X\}.$$

For any $x \in X$ we have $x = e_0^1(x)$, thus $x \in M$ hence $X \subseteq M$. Also, from the corollary 3.b), it follows $M \subseteq \langle X \rangle$.

We will prove now that $M \in S(\mathfrak{A})$. We take $\gamma < o(\tau)$ and $c_0, ..., c_{n_\gamma - 1} \in M$. It means that there exist $m_i \in \mathbb{N}$, $p_i \in P^{(m_i)}(\mathfrak{P}^*(A))$, $x_0^i, ..., x_{m_i - 1}^i \in X$, $i \in \{0, ..., n_\gamma - 1\}$ such that $c_i \in p_i(x_0^i, ..., x_{m_i-1}^i)$, $\forall i \in \{0, ..., n_\gamma - 1\}$. According to the corollary 8.2 from [4], for any n-ary term function p over $\mathfrak{P}^*(A)$ and for any $m \geq n$ it exists a m-ary term function q over $\mathfrak{P}^*(A)$ such that $p(A_0, ..., A_{n-1}) = q(A_0, ..., A_{m-1})$, for all $A_0, ..., A_{m-1} \in P^*(A)$; this allows us to consider instead of $p_0, ..., p_{n_\gamma - 1}$ the term functions $q_0, ..., q_{n_\gamma - 1}$ all with the same arity $m = m_0 + ... + m_{n_\gamma - 1}$ and the elements $y_0, ..., y_{m-1} \in X$ (which are the elements $x_0^0, ..., x_{m_0-1}^0, ..., x_0^{n_\gamma - 1}, ..., x_{m_{n_\gamma - 1}-1}^{n_\gamma - 1}$) such that $c_i \in q_i(y_0, ..., y_{m-1})$, $\forall i \in \{0, ..., n_\gamma - 1\}$. Then:

$$\begin{aligned} f_\gamma(c_0, ..., c_{n_\gamma-1}) \subseteq f_\gamma(q_0(y_0, ..., y_{m-1}), ..., q_{n_\gamma-1}(y_0, ..., y_{m-1})) = \\ = f_\gamma(q_0, ..., q_{n_\gamma-1})(y_0, ..., y_{m-1}), \end{aligned}$$

and because $f_\gamma(q_0, ..., q_{n_\gamma-1}) \in P^{(m)}(\mathfrak{P}^*(A))$ $(m \in \mathbb{N})$, $y_0, ..., y_{m-1} \in X$ it results $f_\gamma(c_0, ..., c_{n_\gamma-1}) \subseteq M$.

5. Remarks. a) If $\mathfrak{A}$ has nullary multioperations then $\langle \emptyset \rangle = \langle A_0 \rangle$ is the union of the sets which are algebraic constants for $\mathfrak{P}^*(A)$.

b) The constructions of the generated subhypergroupoid and of the generated subsemihypergroup are immediate now from the theorem 4.

c) We can observe that a hypergroup H is a multialgebra with three binary multioperations $(H, \circ, /, \backslash)$, with "$\circ$" associative, where for all $a, b \in H$ we consider $a/b = \{x \in H |\ a \in x \circ b\}$ and $b \backslash a = \{x \in H |\ a \in b \circ x\}$. The construction of the term functions obtained in a hypergroup using the multioperations "$/$" and "$\backslash$" is not easy; although the construction of the generated hypergroup can be made from the theorem 1., and it results the theorem 18, cap. III, from [2]: if $(H, \circ)$ is a hypergroup and $X \neq \emptyset$, $X \subseteq H$ then $\langle XX \rangle = \bigcup_{k \in \mathbb{N}} X_k$, where $X_0 = X$ and $X_{k+1} = X_n \cup (X_n \circ X_n) \cup (X_n / X_n) \cup (X_n \backslash X_n)$; it is also justified the equality $\langle \emptyset \rangle = \emptyset$.

Recall that if $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)_{\gamma < o(\tau)}$ and $\mathfrak{B} = (B, f_0, f_1, ..., f_\gamma, ...)_{\gamma < o(\tau)}$ are multialgebras then a map $h : A \to B$ is called ideal homomorphism if for any $\gamma < o(\tau)$ we have:

$$h(f_\gamma(a_0, ..., a_{n_\gamma-1})) = f_\gamma(h(a_0), ..., h(a_{n_\gamma-1})),\ \forall\ a_0, ..., a_{n_\gamma-1} \in A.$$

In this paper we will work only with this kind of homomorphisms.

If φ is an equivalence on A we say that φ is ideal on $\mathfrak{A}$ if for any $\gamma < o(\tau)$ we have:

$$a \in f_\gamma(x_0, ..., x_{n_\gamma-1}) \text{ and } x_i \varphi y_i,\ \forall i \in \{0, ..., n_\gamma - 1\} \ \Rightarrow\ \exists b \in f_\gamma(y_0, ..., y_{n_\gamma-1}) \text{ with } a \varphi b.$$

(For instance the regular equivalences of a hypergroupoid are ideal equivalences on it.)

Pickett presents in [6] this:

6. Theorem. *If $h : A \to B$ is an ideal homomorphism between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ then the relation $\varphi = \{(x, y) \in A^2 |\ h(x) = h(y)\}$ is an ideal equivalence on $\mathfrak{A}$. Conversely, if φ is an ideal equivalence on $\mathfrak{A}$, then $p = p_\varphi : A \to A/\varphi$ is an ideal homomorphism. Moreover, the map $b \mapsto p(h^{-1}(b))$ is an isomorphism from $h(\mathfrak{B})$ onto $\mathfrak{A}/\varphi$.*

7. Remark. Suppose that h is an homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are multialgebras of the same type. We construct the algebras $\mathfrak{P}^*(\mathfrak{A})$ and $\mathfrak{P}^*(\mathfrak{B})$. The homomorphism h induce an map $h' : P^*(\mathfrak{A}) \to P^*(\mathfrak{B}),\ h'(X) = \{h(x)|\ x \in X\},\ \forall X \subseteq A,\ X \neq \emptyset$.

Let us consider A a set and $P^*(A)$ the set of its nonvoid subsets. Let φ be a equivalence relation on A and let us consider the relation $\overline{\varphi}$ defined on $P^*(A)$ as follows:

$$A \overline{\varphi} B \ \Leftrightarrow\ \forall\ a \in A,\ \exists\ b \in B \text{ such that } a\varphi b \text{ and } \forall\ b \in B,\ \exists\ a \in A \text{ such that } a\varphi b.$$

It is immediate that $\overline{\varphi}$ is an equivalence on $P^*(A)$. We will prove the following:

8. Theorem. *An equivalence φ on a multialgebra $\mathfrak{A}$ is ideal if and only if $\overline{\varphi}$ is a congruence on $\mathfrak{P}^*(\mathfrak{A})$.*

Proof. Let us suppose that φ is an ideal equivalence on $\mathfrak{A}$ and let us consider $\gamma < o(\tau)$ and $X_i, Y_i \subseteq A$ nonvoids $(i \in \{0, ..., n_\gamma - 1\})$ such that $X_i \overline{\varphi} Y_i,\ \forall\ i \in \{0, ..., n_\gamma - 1\}$. Then, for any $a \in f_\gamma(X_0, ..., X_{n_\gamma - 1})$, it exists $x_0 \in X_0, ..., x_{n_\gamma - 1} \in X_{n_\gamma - 1}$ such that $a \in f_\gamma(x_0, ..., x_{n_\gamma - 1})$; from the definition of $\overline{\varphi}$ it results the existence of $y_0 \in \in Y_0, ..., y_{n_\gamma - 1} \in Y_{n_\gamma - 1}$ with $x_i \varphi y_i,\ \forall\, i \in \{0, ..., n_\gamma - 1\}$, and because φ is ideal, it exists $b \in \in f_\gamma(y_0, ..., y_{n_\gamma - 1}) \subseteq f_\gamma(Y_0, ..., Y_{n_\gamma - 1})$ such that $a\varphi b$. Analogously we can prove that $\forall\ b \in \in f_\gamma(Y_0, ..., Y_{n_\gamma - 1}),\ \exists\ a \in f_\gamma(X_0, ..., X_{n_\gamma - 1})$ such that $a\varphi b$. Hence we have that $f_\gamma(X_0, ..., X_{n_\gamma - 1}) \overline{\varphi} f_\gamma(Y_0, ..., Y_{n_\gamma - 1})$ and we have proved $\overline{\varphi}$ is a congruence on $\mathfrak{P}^*(\mathfrak{A})$.

Conversely, let us take $\gamma < o(\tau)$ and $a, x_i, y_i \in A$ $(i \in \{0, ..., n_\gamma - 1\})$ such that $a \in f_\gamma(x_0, ..., x_{n_\gamma - 1})$ and $x_i \varphi y_i,\ \forall\ i \in \{0, ..., n_\gamma - 1\}$. Then, obviously $\{x_i\} \overline{\varphi} \{y_i\},\ \forall\ i \in \in \{0, ..., n_\gamma - 1\}$, and because $\overline{\varphi}$ is a congruence on $\mathfrak{P}^*(\mathfrak{A})$, we can write $f_\gamma(\{x_0\}, ..., \{x_{n_\gamma - 1}\}) \overline{\varphi} f_\gamma(\{y_0\}, ..., \{y_{n_\gamma - 1}\})$, which leads us to the existence of $b \in \in f_\gamma(y_0, ..., y_{n_\gamma - 1})$ for which $a\varphi b$. In this way we have rich the conclusion that φ is ideal and the theorem is proved.

9. Corollaries. a) If $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ is a multialgebra, φ is an ideal equivalence on $\mathfrak{A}$ and $p \in P^{(n)}(\mathfrak{P}^*(A))$, then for any $\gamma < o(\tau)$ and for any $X_i, Y_i \subseteq A$ nonvoids, with $X_i\ \overline{\varphi}\ Y_i\, (i \in \{0, ..., n_\gamma - 1\})$ we have $p(X_0, ..., X_{n-1})\ \overline{\varphi}\ p(Y_0, ..., Y_{n-1})$.

b) If $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ is a multialgebra, φ is an ideal equivalence on $\mathfrak{A}$ and $p \in \in P^{(n)}(\mathfrak{P}^*(A))$, then for any $\gamma < o(\tau)$ and for any $x_i, y_i \in A$ with $x_i\ \varphi\ y_i\, (i \in \{0, ..., n_\gamma - -1\})$we have $p(x_0, ..., x_{n-1})\ \overline{\varphi}\ p(y_0, ..., y_{n-1})$.

Let h be an homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ and let us take $\varphi = \{(x, y) \in A^2 |\ h(x) = = h(y)\}$. Then we have $\overline{\varphi} = \{(X, Y) \in (P^*(\mathfrak{A}))^2 |\ h'(X) = h'(Y)\}$. Obviously, φ is ideal for $\mathfrak{A}$ if and only if $\overline{\varphi}$ is a congruence on $\mathfrak{P}^*(\mathfrak{A})$. So we can find again a theorem from [6]:

10. Theorem. *The map h' is an homomorphism between the universal algebras $\mathfrak{P}^*(\mathfrak{A})$ and $\mathfrak{P}^*(\mathfrak{B})$ if and only if h is an ideal homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.*

11. Corollaries. a) Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ and $\mathfrak{B} = (B, f_0, f_1, ..., f_\gamma, ...)$ be two multialgebras of the same type τ, $h : A \to B$ an homomorphism and $p \in \in P^{(n)}(\mathfrak{P}^*(A))$. Then for all $A_0, ..., A_{n-1} \subseteq A$ nonvoid parts we have $h'(p(A_0, ..., A_{n-1})) = = p(h'(A_0), ..., h'(A_{n-1}))$.

b) Let $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ and $\mathfrak{B} = (B, f_0, f_1, ..., f_\gamma, ...)$ two multialgebras of the same type τ, $h : A \to B$ an homomorphism and $p \in P^{(n)}(\mathfrak{P}^*(A))$. Then for all $a_0, ..., a_{n-1} \in A$ we have $h'(p(a_0, ..., a_{n-1})) = p(h(a_0), ..., h(a_{n-1}))$.

12. Remarks. a) Let us remember that for a given multialgebra $\mathfrak{A}$ and for a given equivalence ρ on A, A/ρ can be seen as a multialgebra $\mathfrak{A}/\rho$ with the multioperations:

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma -1}\rangle) = \{\rho\langle b\rangle|\ b \in f_\gamma(b_0, ..., b_{n_\gamma -1}),\quad b_i \in \rho\langle a_i\rangle,\ \ \forall i \in \{0, ..., n_\gamma - 1\}\},\quad \gamma < o(\tau) \tag{1}$$

and that we write for $X, Y \subseteq A$, $X\ \overline{\overline{\rho}}\ Y$ if and only if $\forall x \in X$, $\forall y \in Y$, $x\rho y$. The equivalence relations ρ on A for which $\mathfrak{A}/\rho$ is a universal algebra are those equivalences on A which satisfy the following property: if $a, b \in A$ such that $a\rho b$ then for every $\gamma < o(\tau)$ and $x_0, ..., x_{n_\gamma -1} \in A$ we have

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma -1})\ \overline{\overline{\rho}}\ f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma -1}),$$

for all $i \in \{0, ..., n_\gamma - 1\}$.

Indeed, if $a\rho b$, knowing that $x_j \rho x_j$ for all $j \in \{0, ..., n_\gamma - 1\}$, because any f_γ defined on A/ρ by (1) is an operation it results that

$$f_\gamma(x_0, ..., x_{i-1}, a, x_{i+1}, ..., x_{n_\gamma -1})\ \overline{\overline{\rho}}\ f_\gamma(x_0, ..., x_{i-1}, b, x_{i+1}, ..., x_{n_\gamma -1}),$$

for all $i \in \{0, ..., n_\gamma - 1\}$. To prove the converse implication we will proceed as follows: we take any $x, y \in A$ such that $\rho\langle x\rangle, \rho\langle y\rangle \in f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma -1}\rangle)$, with $a_0, ..., a_{n_\gamma -1} \in A$, $\gamma < o(\tau)$; this means that there exist $x_0, ..., x_{n_\gamma -1}, y_0, ..., y_{n_\gamma -1} \in A$ such that $x \in f_\gamma(x_0, ..., x_{n_\gamma -1})$, $y \in f_\gamma(y_0, ..., y_{n_\gamma -1})$ and $x_i \in \rho\langle a_i\rangle$, $y_i \in \rho\langle a_i\rangle$ for all $i \in \{0, ..., n_\gamma - 1\}$. It results that $x_i \rho y_i$, for all $i \in \{0, ..., n_\gamma - 1\}$. We have the following relations:

$$f_\gamma(x_0, x_1, ..., x_{n_\gamma -1})\ \overline{\overline{\rho}}\ f_\gamma(y_0, x_1, ..., x_{n_\gamma -1}),$$

$$f_\gamma(y_0, x_1, x_2, ..., x_{n_\gamma -1})\ \overline{\overline{\rho}}\ f_\gamma(y_0, y_1, x_2, ..., x_{n_\gamma -1}),$$

$$...$$

$$f_\gamma(y_0, ..., y_{n_\gamma -2}, x_{n_\gamma -1})\ \overline{\overline{\rho}}\ f_\gamma(y_0, ..., y_{n_\gamma -2}, y_{n_\gamma -1}),$$

which leads us (from the definition of $\overline{\overline{\rho}}$) to $x\rho y$,i.e. $\rho\langle x\rangle = \rho\langle y\rangle$. This means that f_γ given in (1) is an operation on A/ρ, for any $\gamma < o(\tau)$, and $\mathfrak{A}/\rho$ is a universal algebra.

b) If we are in the case of the remark a) we can define the operations of the universal algebra $\mathfrak{A}/\rho$ as follows:

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma -1}\rangle) = \{\rho\langle b\rangle|\ b \in f_\gamma(a_0, ..., a_{n_\gamma -1})\}. \tag{2}$$

Moreover, we can write

$$f_\gamma(\rho\langle a_0\rangle, ..., \rho\langle a_{n_\gamma -1}\rangle) = \rho\langle b\rangle,\ b \in f_\gamma(a_0, ..., a_{n_\gamma -1}). \tag{3}$$

13. Theorem. *Let* $\mathfrak{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ *be a multialgebra,* ρ *an equivalence on* $\mathfrak{A}$ *for which* $\mathfrak{A}/\rho$ *is a universal algebra and* $p \in P^{(n)}(\mathfrak{P}^*(A))$*, then for any* $\gamma < o(\tau)$ *and for every* $x_i, y_i \in A$ *with* $x_i\ \rho\ y_i\ (i \in \{0, ..., n_\gamma - 1\})$ *we have* $p(x_0, ..., x_{n-1})\ \overline{\overline{\rho}}\ p(y_0, ..., y_{n-1})$.

Proof. We will consider $p = e_i^n$, with $i \in \{0, ..., n-1\}$. Then $p(x_0, ..., x_{n-1}) = e_i^n(x_0, ..., x_{n-1}) = x_i$, and $p(y_0, ..., y_{n-1}) = e_i^n(y_0, ..., y_{n-1}) = y_i$, so $p(x_0, ..., x_{n-1})\ \overline{\overline{\rho}}\ p(y_0, ..., y_{n-1})$.

We suppose that the statement holds for the term functions $p_0, ..., p_{n_\gamma - 1} \in P^{(n)}(\mathfrak{P}^*(A))$ and we will prove it for the term function $p = f_\gamma(p_0, ..., p_{n_\gamma-1})$ (for any $\gamma < o(\tau)$). For every $x \in p(x_0, ..., x_{n-1}) = f_\gamma(p_0(x_0, ..., x_{n-1}), ..., p_{n_\gamma-1}(x_0, ..., x_{n-1}))$ and every $y \in$ $\in p(y_0, ..., y_{n-1}) = f_\gamma(p_0(y_0, ..., y_{n-1}), ..., p_{n_\gamma-1}(y_0, ..., y_{n-1}))$ there exist $a_i \in p_i(x_0, ..., x_{n-1})$ and $b_i \in p_i(y_0, ..., y_{n-1})$ (with $i \in \{0, ..., n_\gamma - 1\}$) such that $x \in f_\gamma(a_0, ..., a_{n_\gamma-1})$, $y \in$ $\in f_\gamma(b_0, ..., b_{n_\gamma-1})$. From the hypothesis of this induction we have $a_i \rho b_i$, $\forall\, i \in \{0, ..., n_\gamma - 1\}$, which leads us, in the same way as in the remark 12.a) to $f_\gamma(a_0, ..., a_{n_\gamma-1})\ \overline{\overline{\rho}}\ f_\gamma(b_0, ..., b_{n_\gamma-1})$, thus we have $x \rho y$, and the theorem is proved.

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"Babeş-Bolyai" University, Department of Mathematics and Computer Science, Str. Mihail Kogălniceanu nr. 1, RO-3400 Cluj-Napoca, ROMANIA