Chapter 1. Modules

1.1. Basic definitions

Rings are throughout with 1 and ring homomorphisms are unitary.

A *left* (resp. *right*) *R-module*, where R is a ring, is an abelian group $(M,+)$ together with a multiplication with scalars

$$R\times M \to M,\ (\alpha, x)\mapsto \alpha x \quad (\text{resp } M\times R\to M,\ (x,\alpha)\mapsto x\alpha)$$

satisfying the axioms:

$$\begin{aligned} \alpha(x+y) &= \alpha x+\alpha y \\ (\alpha+\beta)x &= \alpha x+\beta x \\ (\alpha\beta)x &= \alpha(\beta x) \\ 1\cdot x &= x \end{aligned} \qquad \left(\text{resp. } \begin{aligned} (x+y)\alpha &= x\alpha+y\alpha \\ x(\alpha+\beta) &= x\alpha+x\beta \\ x(\alpha\beta) &= (x\alpha)\beta \\ x\cdot 1 &= x \end{aligned}\right)$$

for all $\alpha,\beta\in R$ and all $x,y\in M$. By an R-module we shall mean a left R-module.

Let R and S be two rings. An (R,S)-bimodule is an abelian group $(M,+)$ such that M is a left R-module, a right S-module and $(\alpha x)\sigma = \alpha(x\sigma)$ for all $\alpha\in R$, $\sigma\in S$, $x\in M$. Example R is a (R,R)-bimodule.

Notations: ${}_RM$, M_R, ${}_RM_S$ (= left, right resp. bimodule)

A nonempty subset N of an R-module M is called a *submodule* if it is closed under both operations. We denote this by

$N\leq M$ (or $N\leq_R M$).

A map $f: M\to M'$ between two R-modules is called R-*linear* (or *homomorphism* of R-modules), provided that

$$f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$$

for all $\alpha,\beta\in R$, $x,y\in M$.

For an R-linear map $f: M'\to M$ we set

$\operatorname{Ker} f = \{x\in M \mid f(x)=0\}$ (Note that $0\in\operatorname{Ker} f$ for all f)

$\operatorname{Im} f = \{y\in M' \mid \exists x\in M: f(x)=y\}$

and we call them the *kernel* respectively the *image* of f

Proposition 1.1.1 Let ${}_R M$ and $N \subseteq M$. Then

$$N \le_R M \quad \text{iff} \quad \begin{cases} N \neq \emptyset \\ \alpha, \beta \in R,\ x, y \in N \Rightarrow \alpha x + \beta y \in N \end{cases}$$

Proof (exercise 1)

Proposition 1.1.2 Let $f: M \to M'$ be an R-linear map. Then $\operatorname{Ker} f \le_R M$ and $\operatorname{Im} f \le_R M'$.

Proof (exercise 2)

Observe that if R is a field then an R-module is nothing but an R-vector space. Note also that the concepts of „abelian group" and „$\mathbb{Z}$-module" coincide.

An obvious example of a linear map is the identity $1_M: M \to M$ for every R-module M. Two R-modules M and M' are called isomorphic if there are R-linear maps $\alpha: M \to M'$ and $\beta: M' \to M$ such that $\beta \cdot \alpha = 1_M$ and $\alpha \cdot \beta = 1_{M'}$. The linear maps α and β are called isomorphisms.

Lemma 1.1.3 A linear map is an isomorphism iff it is bijective.

Proof (exercise 3)

Proposition 1.1.4 If $N \le_R M$, then the relation $x \sim y$ iff $x - y \in N$ is an equivalence relation on M and the factor set is

$$M/_N := \{x + N \mid x \in N\}$$

Moreover the operations

$$+ : M/N \times M/N \to M/N \quad , \quad (x+N) + (y+N) = (x+y) + N$$

$$\cdot : R \times M/N \to M/N \quad , \quad \alpha\,(x+N) = \alpha x + N$$

are well defined and make M/N into an R-module.

In fact on the factor group M/N we can define a multiplication with salars making it into an R-module.

~~Proof~~. The relation $\sim$ is obvious reflexive, transitive and symmetric thus it is an equivalence. Moreover if $x \in M$ its equivalence class is $\{y \in M / y \sim x\} = \{y \in M / y - x \in N\} = \{y \in M / y \in x + N\} = x + N$
$= \{x + m / m \in N\}$

Therefore $M/\sim = M/N = \{x + N / x \in M\}$.

Now if $x + N = x' + N$ and $y + N = y' + N$ then $x - x' \in N$, $y - y' \in N$ so $(x+y) - (x'+y') = (x - x') + (y - y') \in N \Rightarrow (x+y) + N = (x'+y') + N$

~~$\alpha \in$ $\alpha x - \alpha x' = \alpha$~~ If $\alpha \in R$ then $\alpha x - \alpha x' = \alpha (x - x') \in N \Rightarrow$ $\alpha x + N = \alpha x' + N$ and the operations on M/N are well defined. Now the verifications that $(M/N, +)$ is an abelian group and that ~~multi~~ the multiplication with scalars makes it into an R-module are straightforward.

If $N \leq_R M$ the module M/N constructed in Prop. 1.1.4 is called the <u>factor module</u> of M through N.

~~Theorem (The first isomorphism theorem). A linear map $f : M \to M'$ induces an isomorphism~~

The map $p_N : M \longrightarrow M/N$, $p_N(x) = x + N$ is obvious surjective and it is R-linear (exercise 4). It is called the <u>canonical projection</u> corresponding to the submodule N of M.

<u>Theorem 1.1.5</u> (The first isomorphism theorem) Let $f : M \to M'$ be an R-linear map. Then $\operatorname{Ker} f \leq M$, $\operatorname{Im} f \leq M'$ and f induces an isomorphism $M/\operatorname{Ker} f \longrightarrow \operatorname{Im} f$ given by

~~$x + \operatorname{Ker} f \mapsto f(x)$~~ $x + \operatorname{Ker} f = f(x)$

<u>Proof</u> $\operatorname{Ker} f \leq_R M$ and $\operatorname{Im} f \leq_R M'$ is Exercise 5.

~~The map $\bar{f} : M/\operatorname{Ker} f \to \operatorname{Im} f$, $\bar{f}(x + \operatorname{Ker} f)\, f(x)$ is well defined~~

The map $\bar{f} : M/\mathrm{Ker} f \longrightarrow \mathrm{Im} f$, $\bar{f}(x + \mathrm{Ker} f) = f(x)$ is well defined. Indeed if $x + \mathrm{Ker} f = x' + \mathrm{Ker} f$ i.e. $x - x' \in \mathrm{Ker} f$ then $f(x - x') = 0$ so $f(x) = f(x')$. It is surjective by construction. Moreover if $\bar{f}(x + \mathrm{Ker} f) = \bar{f}(x' + \mathrm{Ker} f)$ then $f(x) = f(x')$ so $x - x' \in \mathrm{Ker} f$ ~~and~~ and $x + \mathrm{Ker} f = x' + \mathrm{Ker} f$ proving the fact that $\bar{f}$ is injective.

Therefore $\bar{f}$ is an isomorphism. ~~Note that it is (the unique!)~~

Note that $\bar{f}$ makes the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "p_{\mathrm{Ker} f}"'] & M' \\
M/\mathrm{Ker} f \arrow[r, "\bar{f}"] & \mathrm{Im} f \arrow[u, "i_f"']
\end{tikzcd}$$

commutative, where $p_{\mathrm{Ker} f} : M \to M/\mathrm{Ker} f$, $p_{\mathrm{Ker} f}(x) = x + \mathrm{Ker} f$ is the canonical projection and $i_f : \mathrm{Im} f \longrightarrow M'$ $i_f(y) = y$ is the inclusion map (which is obviously R-linear).

A sequence of R-linear maps of the form

$$\cdots \to M^{i-1} \xrightarrow{f^{i-1}} M^i \xrightarrow{f^i} M^{i+1} \to \cdots$$

is called <u>exact at M_i</u> if $\mathrm{Im} f^{i-1} = \mathrm{Ker} f^i$. The sequence is called <u>exact</u> if it is exact at each M^i, $i \in \mathbb{Z}$.

<u>Remark</u> that the inclusion $\mathrm{Im} f^{i-1} \subseteq \mathrm{Ker} f^i$ is equivalent to $f^i f^{i-1} = 0$.

<u>Lemma</u> 1.1.6 A sequence of the form

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact iff f is injective, g is surjective and $M'' \cong M/\mathrm{Im} f$.

<u>Proof</u> Exercise 6.

Note that an exact sequence of the form

$$0 \to M' \to M \to M'' \to 0$$

is called a <u>short exact sequence</u>. We shall also note that an injective R-linear map is also called a <u>monomorphism</u> and a surjective R-linear map is called an <u>epimorphism</u>

Remark also that, if $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact seq. of R-modules and R-linear maps, then M' is isomorphic to an R-submodule of M and M'' is isomorphic to a quotient of M. More precisely $M' \cong \operatorname{Im} f$ and $M'' \cong M/\operatorname{Im} f$. Thus basically a short exact sequence is ~~is~~ a sequence of the form

$$0 \to L \to M \to M/L \to 0$$

the morphisms being the canonical ones.

Proposition 1.1.7 Consider the commutative diagram with exact rows of R-modules and R-linear maps:

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \to & L' & \xrightarrow[\alpha']{} & M' & \xrightarrow[\beta']{} & N' & \to & 0 \end{array}$$

If f and h are monomorphisms (epimorphisms resp. isomorphisms) then so is g as well.

Proof We shall use the following characterisation of monomorph.'s: if $g: M \to M'$ is an R-linear map then g is injective (a monom.) iff $\operatorname{Ker} g = \{0\}$. (Exercise 7). Suppose f and h are monomorph.'s and let $x \in \operatorname{Ker} g$ i.e. $x \in M$ such that $g(x) = 0$. Then $(h \cdot \beta)(x) = (\beta' \cdot g)(x) = \beta'(0) = 0$. Since h is injective it follows $\beta(x) = 0$, so $x \in \operatorname{Ker} \beta = \operatorname{Im} \alpha$. Thus there is $y \in L$ such that $\alpha(y) = x$. We have $(\alpha' \cdot f)(y) = (g \cdot \alpha)(y) = g(x) = 0$ and α', f are both injective, so $y = 0$. Therefore $x = \alpha(0) = 0$, and $\operatorname{Ker} f = 0$. The proof of the statement concerning epimorphisms is Exercise 8 and the correspondent statement concerning isomorphisms follows.

<u>Exercice 9</u> : Consider a ring $(R,+,\cdot)$ and denote by R^{opp} the ring with the same underlying set as R, the addition defined in the same way and the multiplication given by

$$*: R\times R \longrightarrow R \qquad x*y = yx.$$

Show that an abelian group $(M,+)$ is a left R-module iff it is a right R^{opp} module.

<u>Exercice 10</u>. Every ring may be considered as a left or right module over itself. The corresponding submodules are left resp. right ideals.

<u>Exercice 11</u> Let M_R be a right R-module. An R-linear map $\alpha: M\to M$ is called an <u>endomorphism</u>. Denote by $\mathrm{End}_R(M)$ the set of all endomorphisms of M_R. Show that $(\mathrm{End}_R(M),+,\circ)$ is a ring, when „+" is defined by $(\alpha+\beta)(x)=\alpha(x)+\beta(x)$ for all $x\in M$, and „$\circ$" is the composition of maps. Show that M is an $(\mathrm{End}_R(M),R)$-bimodule w.r.t. the multiplication of scalars

$$\mathrm{End}_R(M)\times M \longrightarrow M \ , \ (\alpha,x)\longmapsto \alpha(x).$$

<u>Exercice 12</u> (The second isomorphism theorem) If $L\leq M\leq N$ are submodules then $M/L\leq N/L$ and $(N/L)/(M/L)\cong N/M$.

<u>Exercice 13</u> (The third isomorphism theorem) If L and M are submodules of N then $L+M=\{x+y \mid x\in L, y\in M\}$ is a submodule of N and $(L+M)/M \cong L/L\cap M$. (Show also that $L\cap M$ is a submodule of N!)

~~<u>Exercice 14</u> Consider a diagram of R-modules and R-linear maps~~

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "f"'] & N \\ M' \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

~~a) There is an R-homomorphism $\bar{f}: M' \to N$ s.t. $\bar{f}$~~

<u>Exercice 14</u> Consider the following diagram of R-modules and R-linear maps

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & M' \\ N \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

with p an epimorphism (surjective).

a) There is an R-homomorphism $\bar{f}: N \to M'$ s.t. $\bar{f}\circ p = f$ iff $\ker p \subseteq \ker f$. Moreover in this case $\bar{f}$ is unique.

b) If $\bar{f}$ does exist then $\bar{f}$ is injective (resp. surjective) iff $\ker p = \ker f$ (resp. f is surjective).

<u>Exercice 15</u> (The universal property of the kernel). Let $M \xrightarrow{f} M'$ be an R-linear map and denote $K = \ker f$ and $i: K \to M$ the inclusion ($i(x) = x$ for all $x \in K$). Then $f\circ i = 0$ (the zero map $0: K \to M'$, $0(x) = 0$!) and for every $g: N \to M$ an R-linear map such that $f\circ g = 0$ there is a unique $g': N \to K$ such that $i\circ g' = g$.

$$\begin{tikzcd} N \arrow[d, dashed, "g'"'] \arrow[dr, "g"] & & \\ K \arrow[r, "i"'] & M \arrow[r, "f"] & M' \end{tikzcd}$$

Moreover this universal property determines the kernel up to an isomorphism, that is if $K' \xrightarrow{i'} M$ is an R-linear map such that for every $g: N \to M$ with $f\circ g = 0$ there is a unique $g': N \to K'$ such that $i'\circ g' = g$ then there is a unique isomorphism $k: K' \to K$ such that $i\circ k = i'$.

<u>Exercice 16</u> (The universal property of the image) Let $M \xrightarrow{f} M'$ be an R-linear map and denote by $I = \operatorname{Im} f$ and by $i: I \to M'$ the inclusion. Then there is a unique R-linear map $f': M \to I$ such that $i\circ f' = f$. Remark that f' is surjective. Moreover if $j: N \to M'$ is the inclusion of another submodule of M' such that there is $f'': M \to N$ with $j\circ f'' = f$ then there is a unique R-linear map $g: \operatorname{Im} f \to N$ making commutative the diagram beside.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f'"] \arrow[ddr, bend right, "f''"'] & & M' \\ & \operatorname{Im} f \arrow[ur, "i"] \arrow[d, "g"] & \\ & N \arrow[uur, bend right, "j"'] & \end{tikzcd}$$

Exercise 17 If

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a short exact sequence, then there is an isomorphism $M' \cong \operatorname{Ker} g$. Therefore for every $h : N \to M$ with $gh = 0$ there is a unique $h' : N \to M'$ such that $fh' = h$.

Exercise 18 For every R-linear map $f : M \to N$ there is an exact sequence

$$0 \to \operatorname{Ker} f \to M \xrightarrow{f} N \to N/\operatorname{Im} f \to 0$$

the morphisms being the canonical ones. Note also that $N/\operatorname{Im} f$ is called the cokernel of f and it is denoted by $N/\operatorname{Im} f = \operatorname{Coker} f$.

Exercise 19 The cokernel of an R-homomorphism satisfies the following universal property: If $f : M \to N$ is an R-homomorphism and $p : N \to N/\operatorname{Im} f = \operatorname{Coker} f$ is the canonical projection, then $p \circ f = 0$ and if

Exercise 20. Show that an abelian group $(M, +)$ is a $\mathbb{Z}(n)$-module $(x \in M,\ n \geq 2)$ iff $nx = \underbrace{x + x + \dots + x}_{n\ \text{times}} = 0$ in M.

Exercise 21. Let M be an R-module, and let $X \subseteq M$ be a subset of the underlying set of M.

a) Show that if $\{M_i \mid i \in I\}$ is a family of submodules, then $\bigcap_{i \in I} M_i \in M$.

b) Using as the set $\langle X \rangle = \bigcap \{ N \leq_R M \mid X \subseteq N \}$ is a submodule of M. Actually it is the smallest submodule which contains X. Show that $\langle X \rangle = \{ a_1 x_1 + \dots + a_n x_n \mid a_1, \dots, a_n \in R,\ x_1, \dots, x_n \in X,\ n \geq 0 \}$

Note: $\langle X \rangle$ is called the submodule generated by X. If $\langle X \rangle = M$, then X is called a family of generators for M. An R-module M is called finitely generated if it has a finite family of generators.

1.2. The Group of Homomorphisms

For two R-modules M and N we set

$$\mathrm{Hom}_R(M,N) = \{ f : M \to N \mid f \text{ is } R\text{-linear} \}.$$

Proposition 1.2.1. a) $(\mathrm{Hom}_R(M,N), +)$ is an abelian group, where

$$f+g : M \to N, \quad (f+g)(x) = f(x) + g(x).$$

b) If $_RM_S$ and $_RN_T$ are bimodules, then $\mathrm{Hom}_R(M,N)$ is a (S,T)-bimod.

c) If $_SM_R$ and $_TN_R$ are bimodules, then $\mathrm{Hom}_R(M,N)$ is a (T,S)-bimodule.

Proof a) $f+g$ defined above is R-linear since

$$(f+g)(\alpha x + \beta y) = f(\alpha x+\beta y) + g(\alpha x + \beta y) = \alpha f(x) + \beta f(y) + \alpha g(x) + \beta g(y)$$
$$= \alpha\big(f(x)+g(x)\big) + \beta\big(f(y)+g(y)\big) = \alpha (f+g)(x) + \beta(f+g)(y).$$

The addition is obviously associative and commut., the R-linear map

$$0 : M \to N, \quad 0(x) = 0 \text{ for all } x \in M$$

is neutral element, and any R-linear map $f : M \to N$ has an opposite, the linear map $-f : M \to N$, $(-f)(x) = -f(x)$.

b) We only indicate the multiplication with scalars:

$$S \times \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N), \quad (s,f) \mapsto sf \text{ with } (sf)(x) = f(xs).$$
$$\mathrm{Hom}_R(M,N) \times T \to \mathrm{Hom}_R(M,N), \quad (f,t) \mapsto ft, \text{ with } (ft)(x) = f(x)t.$$

The verifications that $\mathrm{Hom}_R(M,N)$ is a (S,T)-bimodule are easy.

c) Use b) and the ~~equivalen~~ 1-to-1 correspondence between left R-modules and right R^{op} modules (see 1.1. Exercise 9). q.e.d.

If $g : N \to N'$ is an R-linear map, then by definition

$$g^* = \mathrm{Hom}_R(M, g) : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'), \quad g^*(f) = g \circ f$$

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ N \arrow[r, "g"'] & N' \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} N \arrow[r, "g"] \arrow[dr, "h\circ g"'] & N' \arrow[d, "h"] \\ & M \end{tikzcd}$$

$$g_* = \mathrm{Hom}_R(g, M) : \mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N,M), \quad g_*(h) = h \circ g.$$

Lemma 1.2.2 Consider a sequence of R-modules $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ and let M be an R-module. We have

a) $(\beta\alpha)^* = \beta^*\alpha^*$

b) $(\beta\cdot\alpha)_* = \alpha_*\cdot\beta_*$

Proof According to the above definitions we have the induced sequences

$$\mathrm{Hom}_R(M,N') \xrightarrow{\alpha^*} \mathrm{Hom}_R(M,N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M,N'')$$

$$\mathrm{Hom}_R(N'',M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N,M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(N',M)$$

Moreover for $f \in \mathrm{Hom}_R(M,N')$ and $g \in \mathrm{Hom}_R(N'',M)$ we have

$$(\beta^*\cdot\alpha^*)(f) = \beta^*(\alpha^*(f)) = \beta^*(\alpha\cdot f) = \beta\cdot(\alpha\cdot f) = (\beta\alpha)\cdot f = (\beta\cdot\alpha)^*(f)$$

$$(\alpha_*\cdot\beta_*)(g) = \alpha_*(\beta_*(g)) = \alpha_*(g\cdot\beta) = (g\cdot\beta)\cdot\alpha = g\cdot(\beta\cdot\alpha) = (\beta\cdot\alpha)_*(g).$$

Theorem 1.2.3 Let $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \to 0$ be an exact sequence of R-modules.

a) For any R-module M the induced sequence of abelian groups

$$0 \to \mathrm{Hom}_R(M,N') \xrightarrow{\alpha^*} \mathrm{Hom}_R(M,N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M,N'')$$

is exact

b) For any R-module M the induced sequence of abelian groups

$$0 \to \mathrm{Hom}_R(N'',M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N,M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(N',M)$$

is exact.

Proof a) First we show that α^* is injective. Let $f,g : M \to N'$ be linear maps such that $\alpha^*(f) = \alpha^*(g)$. Then $\alpha\cdot f = \alpha\cdot g$ so $f = g$ since α is left cancellable, being injective.

~~For Now $(\beta^*\alpha^*)(f) = \beta\alpha$ $(\beta^*\cdot\alpha^*)(f) = (\beta\cdot\alpha)^*(f) = (\beta\cdot\alpha)\cdot f = 0\cdot f = 0$~~

~~$(\beta^*\cdot\alpha^*)(f) = (\beta\cdot\alpha)\cdot f = 0\cdot f = 0$ so $\beta^*\cdot\alpha^* = 0$~~

Now, for all $f \in \mathrm{Hom}_R(M,N')$ we have

$$(\beta^*\cdot\alpha^*)(f) = (\beta\cdot\alpha)\cdot f = 0\cdot f = 0 \quad \text{so} \quad \mathrm{Im}\,\alpha^* \subseteq \mathrm{Ker}\,\beta^*.$$

Conversely, if $g \in \operatorname{Ker}\beta^*$ that means $g: M \to N$ is an R-linear map such that $\beta^*(g) = 0$ i.e. $\beta \cdot g = 0$, ~~then~~ Thus $\operatorname{Im} g \subseteq \ker\beta = \operatorname{Im}\alpha$. By ~~Exercice 16~~ the universal property of the image (see Ex. 1.1.16) we get a morphism $f: M \to N'$ such that $g = \alpha \cdot f = \alpha^*(f)$ so $g \in \operatorname{Im}\alpha^*$.

b) Exercice 1.

Exercice 2 Show that with the settings of Theorem 1.2.3. the map β^* is not, in general, surjective even β is. (Hint: Take $R = \mathbb{Z}$ and the short exact sequence $0 \to \cancel{\mathbb{Z}(n) \to \mathbb{Z} \to \mathbb{Z}/}$

$$\cancel{0 \to \mathbb{Z} \xrightarrow{n\cdot -} \mathbb{Z} \to \mathbb{Z}/n}$$

$$0 \to n\mathbb{Z} \xrightarrow{\alpha_n} \mathbb{Z} \xrightarrow{\beta_n} \mathbb{Z}/n\mathbb{Z} \to 0$$

and for M take $\mathbb{Z}/n\mathbb{Z}$.

Exercice 3 Show that in the settings of Theorem 1.2.3. the map α_* is not, in general, surjective. (Hint: Take $R = \mathbb{Z}$, the short exact sequence $0 \to n\mathbb{Z} \xrightarrow{\alpha_n} \mathbb{Z} \xrightarrow{\beta_n} \mathbb{Z}/n\mathbb{Z} \to 0$ and for M take $\mathbb{Z}$.

Exercice 4 For any R-modules M and N the homomorphism group $\operatorname{Hom}_R(M,N)$ has a natural structure of $(\operatorname{End}_R(M), \operatorname{End}_R(N))$-bimodule.

Exercice 5 Show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \cong A$ for every abelian group A. Generalization for modules over an arbitrary ring.

1.3. Sums and Products of Modules

Let I be a set and assume we have a family $(M_i)_{i\in I}$ of R-modules indexed by the set I. Consider the cartesian product

$$\prod_{i\in I} M_i = \{(x_i)_{i\in I} \,/\, x_i \in M_i \text{ for all } i\in I\}$$

Then M can be made into a module by defining the operations component-wise:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}, \qquad \alpha (x_i)_{i\in I} = (\alpha x_i)_{i\in I}$$

The module $\prod_{i\in I} M_i$ is called the <u>direct product</u> of modules M_i, $i\in I$.
We set

$$\bigoplus_{i\in I} M_i = \{(x_i)_{i\in I} \in \prod_{i\in I} M_i \,/\, x_i = 0 \text{ for almost all } i\in I\}$$

Convention: "almost all" = "all but a finite number".

<u>Remark</u> 1.3.1 a) $\prod_{i\in I} M_i = \bigoplus_{i\in I} M_i$ iff I is finite (<u>Exercise 1</u>).

b) There is an isomorphism of R-modules

$$\prod_{i\in I} M_i \longrightarrow M, \text{ where}$$

$\cancel{M = \{f \,/\, f : I \to\}}$ $M = \{f : I \to \bigcup_{i\in I} M_i \,/\, f(i) \in M_i \text{ for all } i\in I\}$

with the operation defined ~~compo~~ point-wise, namely

$$(f+g)(i) = f(i) + g(i), \quad (\alpha f)(i) = \alpha f(i) \quad \text{for all } i\in I.$$

(~~[illegible]~~) ~~Under the isomorphism~~ Under this isomorphism the direct sum is maped to the submodule of M:

$$\{f\in M \mid \operatorname{supp}(f) \text{ is finite}\}, \text{ where } \operatorname{supp}(f) = \{i\in I \,/\, f(i)\neq 0\}.$$

(<u>Exercise 2</u>).

~~There are canonical homomorphisms $p_j : \prod_{i\in I} M_i \to M_j$, $p_j((x_i)_{i\in I}) = x_j$ ($j\in I$) and $q_j : M_j \to \prod_{i\in I} M_i$, $q_j(x) = (0, 0, \dots x, \dots)$~~

The homomorphisms $p_j : \prod_{i\in I} M_i \longrightarrow M_j$, $p_j((x_i)_{i\in I}) = x_j$ (for all $j\in I$) are called the <u>canonical projections</u> of the product. They have right inverses namely the homomorphisms

$$q_j : M_j \longrightarrow \prod_{i\in I} M_i, \quad q_j(x) = (\delta_{ij} x)_{i\in I} \text{ when } \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

(the <u>canonical injections</u>)

The same is true for $\bigoplus_{i\in I} M_i$. Namely there are homomorphisms $q_j' : M_j \to \bigoplus_{i\in I} M_i$, $q_j'(x) = (\delta_{ij}x)_{i\in I}$ (Note that q_j' are obtained from q_j by restriction of the codomain). These morphisms are called the <u>canonical injections</u> of the direct sums. They have left inverses, namely the homomorphisms

$$p_j' : \bigoplus_{i\in I} M_i \longrightarrow M_j, \quad p_j'((x_i)_{i\in I}) = x_j \quad \left(\text{Clearly } p_j' = p_j\big|_{\bigoplus_{i\in I} M_i}\right).$$

<u>Notations</u>: If $I = \{1, 2, \ldots, n\}$ then we write

$$M_1 \times M_2 \times \ldots \times M_n = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

If $M_i = M$, $i\in I$ then we write $M^I = \prod_{i\in I} M_i$ and $M^{(I)} = \bigoplus_{i\in I} M_i$.

<u>Theorem</u> 1.3.2 Consider a family of homomorphism of R-modules $\{f_i : M_i \longrightarrow N \mid i\in I\}$ with the same codomain. Then there is a unique R-homomorphism $f : \bigoplus_{i\in I} M_i \longrightarrow N$ such that $f\circ q_j' = f_j$ for all $j\in I$, where q_j' are the canonical inclusions of the direct sum $\bigoplus_{i\in I} M_i$. Moreover this universal property determines $\bigoplus_{i\in I} M_i$ up to a unique isomorphism, that means if M is another R-module together with a family of homomorphisms $\varphi_i : M_i \to M$ such that for every family $\{f_i : M_i \to N \mid i\in I\}$ there is a unique $f : M \to N$ with $f\varphi_i = f_i$ for all $i\in I$ then there is a unique isomorphism $\bigoplus_{i\in I} M_i \xrightarrow{\varphi} M$ such that $\varphi\cdot q_i' = \varphi_i$, $i\in I$.

$$\begin{tikzcd}
M_j \arrow[d, "q_j'"'] \arrow[dr, "f_j"] & \\
\bigoplus_{i\in I} M_i \arrow[r, dashed, "f"'] & N
\end{tikzcd}$$

<u>Proof</u> Define $f : \bigoplus_{i\in I} M_i \longrightarrow N$ as follows: if $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ that is $x_i\in M_i$, $i\in I$ s.t. $x_i = 0$ for almost all $i\in I$, then set

$$\cancel{f(x_i)_{i\in I} = (f_i(x_i))} \qquad f(x_i)_{i\in I} = \sum_{i\in I} f_i(x_i) \quad (\text{this sum is finite!})$$

Then f is an R-linear map. Indeed if $(x_i)_{i\in I}, (y_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ and $\alpha\in R$ then $f\left((x_i)_{i\in I} + (y_i)_{i\in I}\right) = f\left((x_i + y_i)_{i\in I}\right) = \sum_{i\in I} f_i(x_i+y_i) = \sum_{i\in I} f_i(x_i) + \sum_{i\in I} f_i(y_i) = f(x_i)_{i\in I} + f(y_i)_{i\in I}$ and $f(\alpha(x_i)_{i\in I}) = \alpha f(x_i)_{i\in I}$.

Moreover for all $j \in I$ we have $(f \cdot q_j)(x) = f(\delta_{ij}x)_{i\in I} = \sum_{i\in I} f_i(\delta_{ij}x) = f_j(x)$ so $f \cdot q_j = f_j$. If $f' : \bigoplus_{i\in I} M_i \to N$ is R-linear such that $f' q_j = f_j$ for all $j \in I$ then ~~for all~~ let $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$. We have

$$(x_i)_{i\in I} = \sum_{i\in I} (\delta_{ij}x_j)_{i\in I} = \sum_{i\in I} q_i(x_i) \qquad \text{(the sum is again finite!)}$$

so

$$f'((x_i)_{i\in I}) = f'\Big(\sum_{i\in I} q_i(x_i)\Big) = \sum_{i\in I} (f' \cdot q_i)(x_i) = \sum_{i\in I} (f \cdot q_i)(x_i) =$$
$$= f\Big(\sum_{i\in I} q_i(x_i)\Big) = f((x_i)_{i\in I}).$$

Thus $f = f'$.

Consider now M satisfying the same universal property as $\bigoplus_{i\in I} M_i$. By the universal property of $\bigoplus_{i\in I} M_i$ we obtain a homomorphism (unique!) $\varphi : \bigoplus_{i\in I} M_i \to M$ making commutative the diagram:

$$\begin{tikzcd} M_j \arrow[d, "q_j'"'] \arrow[dr, "\varphi_j"] & \\ \bigoplus_{i\in I} M_i \arrow[r, dashed, "\varphi"'] & M \end{tikzcd} \qquad \text{for all } j \in I$$

By the universal property of M we obtain an unique! R-homom. making commutative the diagram

$$\begin{tikzcd} M_j \arrow[d, "\varphi_j"'] \arrow[dr, "q_j'"] & \\ M \arrow[r, dashed, "\psi"] & \bigoplus_{i\in I} M_i \end{tikzcd}$$

Then $\psi \circ \varphi : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i$ and $\varphi \circ \psi : M \to M$. Moreover $(\psi \circ \varphi) \cdot q_j' = \psi \cdot (\varphi \cdot q_j') = \psi \cdot \varphi_j = q_j' = 1_{\bigoplus_{i\in I} M_i} \cdot q_j'$ for all $j \in I$ so $\psi \circ \varphi = 1_{\bigoplus_{i\in I} M_i}$ by the unicity of the R-homomorphism in the ~~main~~ conclusion of the universal property of $\bigoplus_{i\in I} M_i$. Similarly using the unicity from the universal property of M we obtain $(\varphi \circ \psi) \cdot \varphi_j = \varphi \cdot (\psi \cdot \varphi_j) = \varphi \cdot q_j' = \varphi_j = 1_M \cdot \varphi_j$ for all $j \in I \Rightarrow \varphi \circ \psi = 1_M$. Thus φ is an isomorphism and $\psi = \varphi^{-1}$ q.e.d.

Note that the last part of the proof of Theorem 1.3.2. is ~~the~~ a model for all proofs of the fact that a universal property determines a mathematical object up to a unique isomorphism.

<u>Theorem</u> 1.3.3 Consider a family of R-homomorphism $\{f_i : N \to M_i, i\in I\}$ with the same domain. Then there is a unique R-homomorphism $f : N \to \prod_{i\in I} M_i$ such that $p_j \circ f = f_j$ for all $j \in I$. Moreover

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, "f_j"'] & \prod_{i\in I} M_i \arrow[d, "p_j"] \\ & M_j \end{tikzcd}$$

this universal property determines $\prod_{i\in I} M_i$ up to ~~a~~ unique isomorphism of R-modules

<u>Proof</u> <u>Exercise 3</u>.

<u>Corollary</u> 1.3.4 ~~[illegible]~~ Let $M_i, i \in I$ be a family of R-modules and let N be another R-module. We have the isomorphis (of abelian groups)

a) $\mathrm{Hom}_R(\bigoplus_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$

b) $\mathrm{Hom}_R(N, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Hom}_R(N, M_i)$.

<u>Proof</u> We shall show how Theorem 1.3.3. implies b). The proof of a) is <u>Exercise 4</u>.

Define $F : \prod_{i\in I} \mathrm{Hom}_R(N, M_i) \longrightarrow \mathrm{Hom}_R(N, \prod_{i\in I} M_i)$ ~~by~~ as follows

If $(f_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_R(N, M_i)$ that is $f_i : N \to M_i$ Then let $f : N \longrightarrow \prod_{i\in I} M_i$ the unique R-linear map given in Theorem 1.3.3. We put

$$F((f_i)_{i\in I}) = f.$$

Conversely if $f : N \to \prod_{i\in I} M_i$ is R linear define

$$G : \mathrm{Hom}_R(N, \prod_{i\in I} M_i) \longrightarrow \prod_{i\in I} \mathrm{Hom}_R(M, N_i), \quad G(f) = (p_i \circ f)_{i\in I}$$

Clearly $(F \circ G)(f) = F((p_i \circ f)_{i\in I}) = f$ by the unicity of f.

$$(G \circ F)((f_i)_{i\in I}) = G(f) = (p_i \circ f)_{i\in I} = (f_i)_{i\in I}$$

It remains to show that one of F and G are group homomorphism

But $G(f+f') = (p_i \circ (f+f'))_{i\in I} = (p_i \circ f)_{i\in I} + (p_i \circ f')_{i\in I} = G(f) + G(f')$. q.e.d.

Let M be a module and $M_i \leq M$, $i \in I$ a family of submodules.

Define $\sum_{i\in I} M_i = \{ \sum_{i \in I} x_i \in M \mid x_i \in M_i,\ x_i = 0 \text{ for almost all } i \in I \}$.

Lemma 1.3.5 $\sum_{i\in I} M_i$ is a submodule of M. Moreover it is the smallest submodule containing $\bigcup_{i\in I} M_i$.

~~Proof Exercise 5~~

Proof Exercise 5.

~~the inclusions $M_j \to \bigcup_{i\in I} M_i$ $\{M_j \to \sum_{i\in I} M_i \mid j \in I\}$ induce~~

~~a unique homomorphism $\alpha : \bigoplus_{i\in I} M_i \to \sum_{i\in I} M_i$~~

The inclusions $\{M_i \to M \mid i \in I\}$ induce a unique homomorphism $\alpha : \bigoplus_{i\in I} M_i \longrightarrow M$ such that $\alpha(x_i)_{i\in I} = \sum_{i\in I} x_i$ by Theorem 1.3.2.

Clearly $\operatorname{Im}\alpha = \sum_{i\in I} M_i$. If it happens that α is injective then it induces an isomorphism $\bigoplus_{i\in I} M_i \xrightarrow{\cong} \sum_{i\in I} M_i$, and we say that $\sum_{i\in I} M_i$ is the internal direct sum of its submodules M_i.

Proposition 1.3.6. The following are equivalent for a family $M_i \leq M$ of submodules:

(i) $\sum_{i\in I} M_i$ is the internal direct sum of M_i, $i \in I$.

(ii) $M_j \cap \sum_{i \neq j} M_i = 0$ for all $j \in I$.

(iii) For every $x \in \sum_{i\in I} M_i$ the ~~decom~~ writting $x = \sum_{i\in I} x_i$, $x_i \in M_i$ is unique up the order of terms of the sum.

Proof Exercise 6.

Corollary 1.3.7 Let $N, L \leq M$ be submodules. Then $M \cong N \oplus L$ iff $N \cap L = 0$ and $N + L = M$.

If L and N are two modules, there is an obvious exact sequence

$$0 \to L \longrightarrow L \oplus N \longrightarrow N \to 0$$

(the morphisms are the canonical ones)

More generally we say that an exact sequence $0 \to L \to M \to N \to 0$ splits if there is an isomorphism $M \cong L \oplus N$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & L\oplus N \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

commutes.

<u>Proposition</u> 1.3.8 The following properties of an exact sequence

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

are equivalent:

(i) The sequence splits.

(ii) There is a homomorphism $\alpha': M \to L$ such that $\alpha'\alpha = 1_L$.

(iii) There is a homomorphism $\beta': N \to M$ such that $\beta\cdot\beta' = 1_N$.

<u>Proof</u>. Consider the injections $i: L \to L\oplus N$, $j: N \to L\oplus N$ and the projections $p: L\oplus N \to L$, $k: L\oplus N \to N$. ~~The diagram~~

a) $\Rightarrow$ b) and c). The diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, equal] & M \arrow[r, "\beta"] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, shift left, "i"] & L\oplus N \arrow[l, shift left, "p"] \arrow[r, shift left, "k"] & N \arrow[l, shift left, "j"] \arrow[r] & 0 \end{tikzcd}$$

shows what we need.

b) $\Rightarrow$ a). The homomorphisms $L \xleftarrow{\alpha'} M \xrightarrow{\beta} N$ defines a unique $L\oplus N = L\times N \longrightarrow M$ by the universal property of the product. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, equal] & L\oplus N \arrow[r, "k"] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

Then $L\oplus N \to M$ is an isomorphism by Proposition 1.1.7.

c) $\Rightarrow$ a) Exercice 7.

Let X be a subset of an R-module M. We say that M is generated by X if every $m \in M$ can be written $m = \sum_{x\in X} a(x)x$, with $a(x) \in R$ with all but finitely many $a(x) = 0$. If it furthermore is true that the coefficients $\{a(x), x \in X\}$ are uniquely determined by x then X is called a <u>basis</u> of M and M is called <u>free</u>

Remark that if R is a field then M is a vector space and it is always free. But generally this is not true as we may seen from:

<u>Proposition</u> 1.3.9 An R-module M is free if and only if $M \cong R^{(I)}$ for some set I.

(Recall that R is both a right and a left module over itself)

<u>Proof</u> The module $R^{(I)}$ is free having a basis $(e_i)_{i\in I}$ with $e_i = (\delta_{ij})_{j\in I} \in R^{(I)}$. Conversely if M is free with the basis $(x_i)_{i\in I}$ then define $R^{(I)} \longrightarrow M$ by $(a_i)_{i\in I} \longmapsto \sum_{i\in I} a_i x_i$. This is an isomorphism as consequence of the definition of a basis.

<u>Corollary</u> 1.3.10 Every module is a quotient module of a free module.

<u>Proof</u>. Let $\{x_i \mid i\in I\}$ be a set of generators of an arbitrary module M. Such a set does exists: take for example all elements of the underlying set of the R-module M. By def. of a generating set, the homomorphism $A^{(I)} \xrightarrow{\alpha} M$, $(a_i)_{i\in I} \longrightarrow \sum a_i x_i$ is surjective. Thus $M \cong A^{(I)}/\operatorname{Ker}\alpha$ by the first isomorphism theorem.

<u>Exercice</u> 8. Show that $\mathbb{Z}(n) \oplus \mathbb{Z}(m) \cong \mathbb{Z}(m\cdot n)$ if and only if $\gcd(n,m)=1$.

<u>Exercice</u> 9. A short exact sequence

$$0 \to L \to M \to N \to 0$$

is also called an <u>extension</u> of N by L. The split exact sequence $0 \to L \to L\oplus N \to N \to 0$ (which always exists!) is called the <u>trivial extension</u>. Find a non-trivial extension of $\mathbb{Z}(2)$ by $\mathbb{Z}(2)$.

<u>Exercice</u> 10 Show that the intersection of a family of submodules is a submodule. Using that it follows ~~that~~ for every subset X of an R-module M that the subset

$$\langle X\rangle = \cap \{ N \leq_R M \mid X \subseteq N\}$$

is a submodule of M. Show that $\langle X\rangle$ is the submodule generated by X.

Exercise 11. For each set X define

$$F_R(X) = \{ \sum_{x \in X} a(x)\, x \mid a(x) \in R ,\ a(x) = 0 \text{ for almost all } x \in X \}$$

the set of all formal linear combinations of elements of X with coefficients in R. Define the addition of two linear combination and the multiplication with scalars in an obvious way ~~(co~~ (component-wise). Then $F_R(X)$ is a free module$\overbrace{\ }^{\text{with base } X}$. More precisely $F_R(X) \cong R^{(X)}$. $F_R(X)$ is called the free module on the set X.

Exercise 12 (The universal property of the basis of a free module) If X is a basis of a free module M then every map $f : X \to N$ into another R-module N extends uniquely to a linear map $\bar{f} : M \to N$. Moreover this ~~prope~~ universal property of the basis determines up to a unique isomorphism free modules having bases of the same cardinality.

Exercise 13 Let R be a commutative ring and $(G, \cdot)$ be a group. Show that the multiplication of the group G induces a multiplication on the free module $R[G]$ on the set G (see exercise 11) such that $R[G]$ becomes a ring. This ring is called the group ring and plays a central rôle in representation theory of groups.

Exercise 14 Establish the isomorphism of abelian groups

$$(\mathbb{R}, +) \cong (\mathbb{Q}^{(I)}, +)$$

when I is a set of the power of continuum.

1.4. The tensor product

Let L_R and ${}_RM$ be two R-modules L at the right and M at the left. A <u>bilinear map</u> from $L\times M$ to an abelian group A is a map $\varphi: L\times M\to A$ such that

$$\varphi(x+x',y)=\varphi(x,y)+\varphi(x',y)$$
$$\varphi(x,y+y')=\varphi(x,y)+\varphi(x,y')$$
$$\varphi(xa,y)=\varphi(x,ay)$$

for all $x,x'\in L$, $y,y'\in M$, $a\in R$.

<u>Theorem 1.4.1.</u> If L_R and ${}_RM$ are modules then there is an abelian group T together with a bilinear map $\tau: L\times M\to T$ satisfying in addition, the following universal property: For every abelian group A and every bilinear map $\varphi: L\times M\to A$, there is a unique ~~isomorphism and~~ R-homomorphism $\alpha: T\to A$ such that $\alpha\cdot\tau=\varphi$.

$$\begin{tikzcd} L\times M \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & T \arrow[d, dashed, "\alpha"] \\ & A \end{tikzcd}$$

Moreover this universal property determines T up to a unique isomorphism.

<u>Proof</u> Let F be the free $\mathbb{Z}$-module (abelian group) on the set $L\times M$, i.e. ~~$F=\sum_{(x,y)\in L\times M} n(x,y)\, x_i$~~ ~~$F=\{\sum_{(x,y)\in L\times M} n(x,y)(x,y)/$~~

$$F=\Big\{\sum_{(x,y)\in L\times M} n(x,y)(x,y)\ /\ n(x,y)\in L\times M \text{ with } n(x,y)=0 \text{ for almost all } (x,y)\in L\times M$$

Let R be the subgroup of F generates by all the elements of the form

$$(x+x',y)-(x,y)-(x',y)$$
$$(x,y+y')-(x,y)-(x,y')$$
$$(xa,y)-(x,ay).$$

Put $T=F/R$ and $\tau: L\times M\to T$, $\tau(x,y)=\overline{(x,y)}$, where by $\overline{(x,y)}$ we denote the class of (x,y) in F/R i.e. $\overline{(x,y)}=(x,y)+R$

By construction τ is bilinear. If $\varphi : L\times M \to A$ is another bilinear map define $\beta : F \to A$ be the unique group homomorphism (=$\mathbb{Z}$-linear map) extending φ. It exists by the universal property of the basis. Since φ is bilinear it follows that β sends all generators of R into 0. Consequently $R \subseteq \ker\beta$ and β factors through $p_R : F \to F/R$ by Exercice 14 1.1. (~~factori~~ Theorem of factorization through a surjection). So there is $\alpha : T \to A$ a group homomorphism making commut. the diagram

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, "p_R"'] & A \\ T = F/R \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

Clearly $(\alpha\cdot\tau)(x,y) = \alpha(\overline{(x,y)}) = \alpha\left(p_R(x,y)\right) = \beta(x,y) = \varphi(x,y)$ for all $(x,y) \in L\times M$ so $\alpha\cdot\tau = \varphi$. The homomorphism $\underset{= \alpha\cdot p_R}{\beta}$ is uniquely ~~determined by φ since it is~~ determined by its values on the basis $L\times M$ of F so it is uniquely determined by φ. Therefore α itself is uniquely determined by φ since p_R is surject. The uniqueness of (T,τ) with the above universal property is straightforward - <u>Exercice 1</u>.

The group T together with the bilinear map $\tau : L\times M \to T$ is called the <u>tensor product</u> of the modules L and M. It is denoted by $T = L \otimes_R M$. It is an abelian group with generators $\overline{(x,y)} = \tau(x,y) \overset{\text{not}}{=} x\otimes y$, what means every element in $L\otimes_R M$ is a finite sum of the form $\sum_{(x,y)\in L\times M} n(x,y)\, x\otimes y$. The generators are the subject of relations

$$(x+x')\otimes y = x\otimes y + x'\otimes y$$
$$x\otimes(y+y') = x\otimes y + x\otimes y'$$
$$(xa)\otimes y = x\otimes(ay)$$

Note that the writting of an element in $L\otimes_R M$ as a sum above is not unique ($L\otimes_R M$ is not the free abelian group on $L\times M$!)
For simplicity we shall denote $\sum_i n_i(x_i\otimes y_i)$ an element in $L\otimes_R M$ instead $\sum_{(x,y)\in L\times M} n(x,y)(x\otimes y)$. Note that this sum is finite, $n_i\in\mathbb{Z}$ and $x_i\in L$, $y_i\in M$. (Actually we work with an index set I with the same cardinality as $L\times M$).

Proposition 1.4.2 If ${}_S L_R$ and ${}_R M_T$ are bimodules then $L\otimes_R M$ is a (S,T)-bimodule.

Proof For every $\sum_i n_i(x_i\otimes y_i)\in L\otimes_R M$ and every $s\in S$, $t\in T$ it is natural to define $s\cdot\left(\sum_i n_i(x_i\otimes y_i)\right)=\sum_i n_i\left((s\, x_i)\otimes y_i\right)$ respectively $\left(\sum_i n_i(x_i\otimes y_i)\right)t=\sum_i n_i\left(x_i\otimes(y_i t)\right)$. The axioms defining a (S,T)-bimodule are trivially verified, the single problem which remains is if these scalar multiplications are well defined. More precisely, as noted, the writting of an element in $L\otimes_R M$ as a sum above is not unique. In order to prove that the multiplication with scalars does not depend of the choice of this writting (representant) let define for every $s\in S$: $\varphi_s: L\times M\to L\otimes_R M$, $\varphi_s(x,y)=(sx)\otimes y$ for all $(x,y)\in L\times M$.
Then φ_s is a bilinear map, so there is a unique groups homomorphism $\alpha_s: L\otimes_R M\to L\otimes_R M$ such that $\alpha_s(x\otimes y)=(sx)\otimes y$.
We have $\alpha_s\left(\sum_i n_i(x_i\otimes y_i)\right)=\sum_i n_i\left((s x_i)\otimes y_i\right)$ so the multiplication with scalars above is well defined. Similarly we proceed for the multiplication with scalars in T.

Proposition 1.4.3 For every ~~right~~ left R-module M we have an isomorphism $R\otimes_R M\cong M$.

Proof The map $\varphi: R\times M\to M$, $\varphi(a,x)=ax$ is bilinear.

Moreover if $\psi : R \times M \to A$ is another bilinear map into an abelian group G then define $\alpha : M \to A$, $\alpha(x) = \psi(1, x)$. We have $(\alpha \circ \varphi)(a, x) = \alpha(ax) = \psi(1, ax) = \psi(a, x)$ so $\alpha \circ \varphi = \psi$. Thus the pair (M, φ) satisfies the universal property of the tensor product, so $M \cong R \otimes_R M$.

<u>Proposition</u> 1.4.4 If L_i $(i \in I)$ are right R-modules and M is a left R-module, then there is an isomorphism

$$\left(\bigoplus_{i \in I} L_i\right) \otimes_R M \xrightarrow{\cong} \bigoplus_{i \in I} (L_i \otimes_R M)$$

<u>Proof</u> Let $q_j : L_j \to \bigoplus_{i \in I} L_i$ be the inclusion map $(j \in I)$. We define the following bilinear maps:

$$\varphi : \left(\bigoplus_{i \in I} L_i\right) \times M \to \bigoplus_{i \in I} (L_i \otimes_R M), \quad \varphi\left((x_i)_{i \in I}, y\right) = (x_i \otimes y)_{i \in I}$$

$$\psi_j : L_j \times M \to \left(\bigoplus_{i \in I} L_i\right) \otimes_R M, \quad \psi_j(x_j \otimes y) = q_j(x_j) \otimes y$$

Then φ determines a unique group homomorphism

$$\alpha : \left(\bigoplus_{i \in I} L_i\right) \otimes_R M \to \bigoplus_{i \in I} (L_i \otimes_R M) \text{ s.t. } \alpha\left((x_i)_{i \in I} \otimes y\right) = (x_i \otimes y)_{i \in I}$$

and ψ_j determines a unique group homomorphism

$$\beta_j : L_j \otimes_R M \to \left(\bigoplus_{i \in I} L_i\right) \otimes_R M \text{ s.t. } \beta_j(x_j \otimes y) = q_j(x_j) \otimes y$$

Further the homomorphisms $\{\beta_j \mid j \in I\}$ determine a unique homomorphism

$$\beta : \bigoplus_{i \in I} (L_i \otimes_R M) \to \left(\bigoplus_{i \in I} L_i\right) \otimes_R M$$

such that $\beta\left((x_i \otimes y)_{i \in I}\right) = (x_i)_{i \in I} \otimes y$.

Obviously α and β are inverse to each other, so they are isomorphisms.

<u>Corollary</u> 1.4.5 If F is a free right R-module with basis $(x_i)_{i \in I}$ then $F \otimes_R M \cong M^{(I)}$ for any R-module M.

Let ~~...~~ $f: L'_R \to L_R$ and ~~...~~ $g: {}_RM' \to {}_RM$ be homomorphisms of right resp. left R-modules. The map

$$\varphi_{f,g}: L' \times M' \longrightarrow L \otimes_R M, \quad \varphi_{f,g}(x,y) = f(x) \otimes g(y)$$

is bilinear, so there is a unique ~~map~~ group homomorphism denoted by $f \otimes_R g : L' \otimes_R M' \longrightarrow L \otimes_R M$ such that $(f \otimes_R g)(x \otimes y) = f(x) \otimes g(y)$.

We also denote $L \otimes_R g = 1_L \otimes_R g$ and $f \otimes_R M = f \otimes_R 1_M$.

<u>Proposition</u> 1.4.6 ~~If $M' \xrightarrow{f_1} M \xrightarrow{f_2} M''$ and~~

If $L'_R \xrightarrow{f} L_R \xrightarrow{f'} L''_R$ and ${}_RM' \xrightarrow{g} {}_RM \xrightarrow{g'} {}_RM''$ are R-linear maps then $(f' \cdot f) \otimes_R (g' \cdot g) = (f' \otimes_R g') \cdot (f \otimes_R g)$.

<u>Proof</u> Exercise.

<u>Proposition</u> 1.4.7 a) If $0 \to M' \xrightarrow{g} M \xrightarrow{g'} M'' \to 0$ is a short exact sequence of left R-modules and L is a Right R-module, then the induced sequence

$$L \otimes_R M' \xrightarrow{L \otimes_R g} L \otimes_R M \xrightarrow{L \otimes_R g'} L \otimes_R M'' \longrightarrow 0$$

is exact.

b) If $0 \to L' \xrightarrow{f} L \xrightarrow{f'} L'' \to 0$ is a short exact sequence of right R-modules and M is a left R-module, then the induced sequence

$$L' \otimes_R M \xrightarrow{f \otimes_R M} L \otimes_R M \xrightarrow{f' \otimes_R M} L'' \otimes_R M \longrightarrow 0$$

is exact.

<u>Proof</u> a) First we show that $L \otimes_R g'$ is an epimorphism (surjective). Let $x \otimes y'' \in L \otimes_R M''$. Since g' is an epimorphism, we deduce that there is $y \in M$ s.t. $g'(y) = y''$. Thus $(L \otimes_R g')(x \otimes y) = x \otimes y''$. Now if $\sum_i n_i (x_i \otimes y_i'') \in L \otimes_R M''$ it is immediate that

$$(L \otimes_R g)\left(\sum n_i (x_i \otimes y_i)\right) = \sum_i n_i (x_i \otimes y_i''),$$ where $y_i \in M$ are ~~chosen~~ chosen such that $g'(y_i) = y_i''$.

We show now the exactness at $L\otimes_R M$. First $(L\otimes_R g')\circ(L\otimes_R g) =$
$= L\otimes_R (g'\circ g) = L\otimes_R 0 = 0$ so $\mathrm{Im}(L\otimes_R g) \subseteq \mathrm{Ker}(L\otimes_R g')$.
By the theorem of factorizing through surjection (Exercise 14 1.1.) there is a unique homomorphism $h: L\otimes_R M/\mathrm{Im}(L\otimes_R g) \longrightarrow L\otimes_R M''$ making commutative the diagram

$$\begin{tikzcd}
L\otimes_R M \arrow[r, "L\otimes_R g'"] \arrow[d, "\mathrm{can}"'] & L\otimes_R M'' \\
(L\otimes_R M)/\mathrm{Im}(L\otimes_R g) \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

(Here $h\Big(\underbrace{x\otimes y + \mathrm{Im}(L\otimes g)}_{\overset{\text{not}}{=}\ \overline{x\otimes y}}\Big) = x\otimes g'(y)$)

~~Thus~~ In order to construct an inverse for h let

$\varphi: L\times M'' \longrightarrow L\otimes_R M/\mathrm{Im}(L\otimes_R g)$, $\varphi(x,y'') = \overline{x\otimes y}$, with $g'(y)=y''$.

This map is well defined, since for $y_1, y_2 \in L\otimes_R M$ such that $g'(y_1) = y'' = g'(y_2)$ we have ~~$x\otimes y_1 = x\otimes y_1 + \mathrm{Im}$~~ $g'(y_1-y_2) = 0$ so $y_1 - y_2 \in \mathrm{Ker}\, g' = \mathrm{Im}\, g \Rightarrow x\otimes(y_1-y_2) \in \mathrm{Im}(L\otimes_R g) \Rightarrow$
$x\otimes y_1 + \mathrm{Im}(L\otimes_R g) = x\otimes y_2 + \mathrm{Im}(L\otimes_R g)$ or equivalent $\overline{x\otimes y_1} = \overline{x\otimes y_2}$.
Moreover φ is obvious bilinear. Thus there is a unique group homomorphism $k: L\otimes_R M'' \longrightarrow L\otimes_R M/\mathrm{Im}(L\otimes_R g)$ such that $k(x\otimes y'') = \overline{x\otimes y}$, with $g'(y) = y''$. We have
$(h\circ k)(x\otimes y'') = h(\overline{x\otimes y}) = x\otimes g'(y) = x\otimes y''$, $x\otimes y'' \in L\otimes_R M''$
$(k\circ h)(\overline{x\otimes y}) = k(x\otimes g'(y)) = \overline{x\otimes y}$
Therefore $k = h^{-1}$, showing that h is an isomorphism. Conseq. $\mathrm{Im}(L\otimes_R g) = \mathrm{Ker}(L\otimes_R g')$ q.e.d.

We call flat an R-module $_RM$ with the property that $f\otimes_R M$ is injective, whenever $f: L'_R \longrightarrow L_R$ is an injective homomorphism of right R-modules.

<u>Theorem</u> 1.4.8 Let $(y_i)_{i\in I}$ be a family of generators for ${}_RM$ and let $(x_i)_{i\in I}$ a family of elements in L_R with almost all $x_i = 0$. Then $\sum_{i\in I} x_i\otimes y_i$ is zero in $L\otimes_R M$ iff there exists a finite family $(u_j)_{j\in J}$ of elements of L and a family $(a_{ji})_{(j,i)\in J\times I}$ of elements in A such that

a) $a_{ji} = 0$ for almost all (j,i).

b) $\sum_{i\in I} a_{ji} y_i = 0$ for each $j\in J$

c) $x_i = \sum_{j\in J} u_j a_{ji}$ for each $i\in I$.

<u>Proof</u> The sufficiency of the conditions a), b), c) is <u>Exercise 3</u>.

For the necessity, let $\sum_{i\in I} x_i\otimes y_i = 0$. Since $(y_i)_{i\in I}$ generates M we get an exact sequence

$$0 \to K \xrightarrow{f} R^{(I)} \xrightarrow{g} M \to 0$$

where $g(e_i) = y_i$, $i\in I$ (Here $\{e_i \mid i\in I\}$ is the canonical basis of $R^{(I)}$).

By tensoring with L we get an exact sequence

$$L\otimes_R K \xrightarrow{L\otimes f} L\otimes_R R^{(I)} \xrightarrow{L\otimes g} L\otimes_R M \to 0$$

Since $\sum_{i\in I} x_i\otimes y_i = 0$ we deduce $\sum_{i\in I} x_i\otimes e_i \in \operatorname{Ker}(L\otimes g) = \operatorname{Im}(L\otimes f)$, so ~~there are $z_j\in K$, $i\in I$ such that~~ $\sum_{i\in I} x_i\otimes e_i = \sum_{j\in J} u_j\otimes f(z_j)$ for a finite set J and for some $u_j\in L$, $z_j\in K$ $(j\in J)$. Each $f(z_j)$ can be expressed in the canonical basis as $f(z_j) = \sum_{i\in I} a_{ji} e_i$.

Then ~~$0=(g\cdot f)(\sum u_j\otimes$~~ $0 = (g\cdot f)(z_j) = g\left(\sum_{i\in I} a_{ji} e_i\right) = \sum_{i\in I} a_{ji} y_i$ for all $j\in J$.

Moreover

$$\sum_{i\in I} x_i\otimes e_i = \sum_{j\in J} u_j\otimes f(z_j) = \sum_{(i,j)\in I\times J} u_j\otimes a_{ji} e_i \quad \text{in } L\otimes R^{(I)}$$

Under the isomorphism $L\otimes R^{(I)} \cong L^{(I)}$, this gives $x_i = \sum_{j\in J} u_j a_{ji}$.

Theorem 1.4.9 Let consider a right R-module L, a left R-module M and an abelian group G. Then there is a group isomorphism

$$\omega_{L,M,G} : \mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G)$$

which is natural in L, M and G (the precise sense of the term natural will be precised in the proof).

Proof Observe first that if M is a left R-module, then M is a $(R,\mathbb{Z})$-bimodule, so $\mathrm{Hom}_{\mathbb{Z}}(M,G)$ is a right R-module by Prop. 1.2.1. Thus, it makes sense to speak about $\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$.

Let $f \in \mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$. We associate to f the bilinear map $\varphi : L \times M \longrightarrow G$, $\varphi(x,y) = f(x)(y)$. (The fact that φ is indeed bilinear is obvious.) It induces a unique $\mathbb{Z}$-linear map, which will be the image of $\omega_{L,M,G}$ in f, more precisely

$$\cancel{\omega_{L,M,G}(f) : L \longrightarrow \mathrm{Hom}_{\mathbb{Z}}}$$

$$\omega_{L,M,G}(f) : L \otimes_R M \longrightarrow G, \qquad \omega_{L,M,G}(f)(x \otimes y) = f(x)(y).$$

If $f, g \in \mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$ then $\omega_{L,M,G}(f+g)(x \otimes y) =$
$= (f+g)(x)(y) = f(x)(y) + g(x)(y) = \omega_{L,M,G}(f)(x \otimes y) + \omega_{L,M,G}(g)(x \otimes y)$
for all $(x,y) \in L \times M$, so $\omega_{L,M,G}$ is a group homomorphism.

Define $\omega'_{L,M,G} : \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G) \longrightarrow \mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$ by $\omega'_{L,M,G}(h) : L \to \mathrm{Hom}_{\mathbb{Z}}(M,G)$, $\omega'_{L,M,G}(h)(x)(y) = h(x \otimes y)$ for all $h \in \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G)$, all $x \in L$, all $y \in M$. It is routine to check that $\omega'_{L,M,G}(h)(x)$ is a group homomorphism and $\omega'_{L,M,G}(h)$ is an R-linear map for all $h \in \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G)$, all $x \in L$.

Moreover $(\omega'_{L,M,G} \cdot \omega_{L,M,G})(f)(x)(y) = \omega_{L,M,G}(f)(x \otimes y) = f(x)(y)$ for all $(x,y) \in L \times M$ so $(\omega'_{L,M,G} \cdot \omega_{L,M,G})(f) = f$ and similarly $(\omega_{L,M,G} \cdot \omega'_{L,M,G})(h) = h$, where $f \in \mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$ and $h \in \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G)$. ~~Moreover~~ That means $\omega'_{L,M,G} = \omega^{-1}_{L,M,G}$ showing that $\omega_{L,M,G}$ is an isomorphism.

The naturality of $\omega_{L,M,G}$ in L means the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G)) \arrow[r, "\omega_{L,M,G}"] \arrow[d, "\alpha_*"'] & \mathrm{Hom}_{\mathbb{Z}}(L\otimes_R M, G) \arrow[d, "(\alpha\otimes_R M)_*"] \\
\mathrm{Hom}_R(L', \mathrm{Hom}_{\mathbb{Z}}(M,G)) \arrow[r, "\omega_{L',M,G}"] & \mathrm{Hom}_{\mathbb{Z}}(L'\otimes_R M, G)
\end{tikzcd}$$

for every R-linear map $\alpha: L'\to L$. Recall that $\alpha_* = \mathrm{Hom}_R(\alpha, \mathrm{Hom}_{\mathbb{Z}}(M,G))$ given by $\alpha_*(f) = f\cdot\alpha$ and $(\alpha\otimes_R M)_*(h) = h\cdot(\alpha\otimes_R M)$ (see Section 1.2). But clearly, for every $f\in\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G))$:

$\cancel{(\omega_{L',M,G}\cdot\alpha_*)(f) = \omega_{L',M,G}(f\cdot\alpha)}$ ~~so~~ $\cancel{\omega_{L',M,G}(f)}$

$\cancel{(\omega_{L',M,G}\cdot\alpha_*)(f)(x\otimes y) = (f\cdot\alpha)(x)(y)}$ ~~for all~~ $\cancel{(x,y)\in L\times M}$.

$\cancel{((\alpha\otimes_R M)_*\cdot\omega_{L,M,G})(f) = (\alpha\otimes_R M)(\omega_{L,M,G}(f))}$ ~~so~~

$\cancel{(\alpha\otimes_R M)(\omega_{L,M,G}(f))(x\otimes y) =}$

$$(\omega_{L',M,G}\cdot\alpha_*)(f) = \omega_{L',M,G}(f\cdot\alpha) : L'\otimes_R M \longrightarrow G$$

$$[\omega_{L',M,G}(f\cdot\alpha)](x\otimes y) = (f\cdot\alpha)(x)(y) = f(\alpha(x))(y), \text{ for all } (x,y)\in L\times M.$$

$$[(\alpha\otimes_R M)_*\cdot\omega_{L,M,G}](f) = \omega_{L,M,G}(f)\cdot(\alpha\otimes_R M) : L'\otimes_R M\longrightarrow G$$

$$[\omega_{L,M,G}(f)\cdot(\alpha\otimes_R M)](x\otimes y) = \omega_{L,M,G}(f)(\alpha(x)\otimes y) = f(\alpha(x))(y), \text{ for all } (x,y)\in L\times M$$

Therefore $\omega_{L',M,G}\cdot\alpha_* = (\alpha\otimes_R M)_*\cdot\omega_{L,M,G}$.

Similarly we show the naturality of ω in M and G, meaning the commutativity of diagrams

$$\begin{tikzcd}
\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,G)) \arrow[r, "\omega_{L,M,G}"] \arrow[d, "{\mathrm{Hom}_R(L,\mathrm{Hom}_{\mathbb{Z}}(\beta,G))}"'] & \mathrm{Hom}_R(L\otimes_R M, G) \arrow[d, "(L\otimes_R\beta)_*"] \\
\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M',G)) \arrow[r, "\omega_{L,M',G}"] & \mathrm{Hom}_R(L\otimes_R M', G)
\end{tikzcd}$$

for all $\beta: {}_RM'\to {}_RM$, respectively

$$
\begin{tikzcd}
\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M, G')) \arrow[r, "\omega_{L,M,G'}"] \arrow[d, "\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M,\gamma))"] & \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G) \arrow[d, "\gamma^* = \mathrm{Hom}_{\mathbb{Z}}(L\otimes_R M, \gamma)"] \\
\mathrm{Hom}_R(L, \mathrm{Hom}_{\mathbb{Z}}(M, G)) \arrow[r, "\omega_{L,M,G}"'] & \mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, G)
\end{tikzcd}
$$

for all $\gamma : G' \to G$ a group homomorphism.

Remark 1.4.10 The same argument shows that if L is a right R-module, M is a (R,S)-bimodule and N is a left R-module, we have ~~an group isomorphism of~~ group homomorphism

$$\mathrm{Hom}_R(L, \mathrm{Hom}_S(M,N)) \longrightarrow \mathrm{Hom}_S(L \otimes_R M, N)$$

which is natural in L, M and N.

~~Exercise 3 Show that there are $\mathbb{Z}$-modules (= abelian groups) which are not flat over $\mathbb{Z}$. Hint: Show that $\mathbb{Z}(p) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ ($p \in \mathbb{N}$ is prime) and $\mathbb{Z}(n) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ for all $n \in \mathbb{Z}$,~~

Exercise 3 Show that for every $n \in \mathbb{N}$, $n \geq 2$ we have $\mathbb{Z}(n) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Deduce that $\mathbb{Z}(n)$ is not flat as $\mathbb{Z}$-module (= abelian groups) by tensoring the injection $\mathbb{Z} \longrightarrow \mathbb{Q}$ with $\mathbb{Z}(n)$.

Exercise 4 If M is an abelian group, then prove the equality

$$\mathbb{Z}(n) \otimes_{\mathbb{Z}} M = M/nM,$$

where $nM = \{ nx \mid x \in M \}$. More generally if I is a right ideal of a ring R (i.e. I is a submodule of R_R), then prove $(R/I) \otimes_R M = M/IM$, where $IM = \{ ax \mid a \in I, x \in M \}$, for every left R-module M.

Exercise 5 Let R, S be rings. Show that the notions "(R,S)-bimodule" and "left $R \otimes_{\mathbb{Z}} S^{op}$-module" coincide.

Exercise 6 Let M be a (R,S)-bimodule. Show that for every right R-module X and every right S-module Y, the applications $\theta_X : X \longrightarrow \mathrm{Hom}_S(M, X \otimes_R M)$, $\theta_X(x) : m \mapsto x \otimes m$ respectively $\rho_Y : \mathrm{Hom}_S(M,Y) \otimes_R M \longrightarrow Y$, $\rho_Y(f \otimes m) = f(m)$ are S-linear, resp. R-linear. Moreover θ_X is natural in X, ρ_Y is natural in Y, and it holds: $\mathrm{Hom}_S(M, \mathcal{J}_Y) \cdot \theta_{\mathrm{Hom}_S(M,Y)} = 1_{\mathrm{Hom}_S(M,Y)}$ and

$$\mathcal{J}_{X \otimes_R M} \cdot (\theta_X \otimes_R M) = 1_{X \otimes_R M}$$

~~for all~~ for every right S-module X and every right R-module Y.

Exercice 7 If R is a commutative ring, then prove that there is a isomorphism of R-modules $M \otimes_R N \cong N \otimes_R M$ for every two R-modules M and N. Moreover this isomorphism is natural both in M and N.

Exercice 8 If L_R, ${}_RM_S$, ${}_SN$ are modules, show that there is a group ~~isomorphism~~ $(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$ which is natural in L, M and N. This isomorphism allows us to write $L \otimes_R M \otimes_S N$.

Exercice 9 Let R be a commutative ring. Show that the tensor product $M \otimes_R M \otimes_R \dots \otimes_R M$ with n factors may be characterized by the following universal property: ~~for every multilinear~~ the ~~application~~ map $M \times M \times \dots \times M \xrightarrow{\varphi} M \otimes_R M \otimes_R \dots \otimes_R M$, i

$\varphi : (x_1, x_2, \dots, x_n) \mapsto x_1 \otimes x_2 \otimes \dots \otimes x_n$ is multilinear, and for every multilinear map $\psi : M \times M \times \dots \times M \to \cancel{N}$ into ~~an abelian group G~~ an R-module N, there is a unique R-linear map ~~$f: M \otimes_R \dots \otimes_R M \to$~~ $f : M \otimes_R M \otimes_R \dots \otimes_R M \to N$ such that $f \cdot \varphi = \psi$.

Here ψ is called multilinear if it is linear in each variable.

1.5. Algebras

In this section consider a commutative ring R. An R-algebra is a structure consisting of an R-module A together with a ring structure $(A,+,\cdot)$ (recall: with 1) such that for ~~every~~ all $r \in R$ and all $a,b \in A$ we have

$$r(ab) = (ra)b = a(rb).$$

~~Every ring is a $\mathbb{Z}$-algebra.~~

A homomorphism of R-algebras is a unitary ring homomorphism which is also R-linear, that is a map $f: A \to A'$ between two R-algebras satisfying

$$f(a+b) = f(a) + f(b)$$
$$f(ab) = f(a)f(b)$$
$$f(1) = 1$$
$$f(ra) = rf(a)$$

for all $a,b \in A$ and all $r \in R$.

Note that every ring is a $\mathbb{Z}$-algebra and every unitary ring homomorph. is a homomorphism of $\mathbb{Z}$-algebras

Lemma 1.5.1 a) If $(A,+,\cdot)$ is a ring then

$$Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$$

is a subring of A containing 1, called the center of A.

b) Giving an R-algebra structure on the ring $(A,+,\cdot)$ is equivalent to give a unitary ring homomorphism $\varphi: R \to A$ such that $\operatorname{Im}\varphi \subseteq Z(A)$. Moreover if $\varphi: R \to A$, $\varphi': R \to A'$ are two unitary ring homomorphisms such that $\operatorname{Im}\varphi \subseteq Z(A)$ and $\operatorname{Im}\varphi' \subseteq Z(A')$ then a ring homomorphism $f: A \to A'$ is a homomorphism of R-algebras iff $f \circ \varphi = \varphi'$.

Proof Exercise 1.

Examples 1.5.2 If R is a commutative ring then

a) $(M_n(R),+,\cdot)$ is an R-algebra (Exercise 2)

b) $(\operatorname{End}_R(M),+,\circ)$ is an R-algebra, for every R-module M (Exercise 3).

c) $(R[X],+,\cdot)$ is a free R-algebra, what means $R[X]$ is a free R-module with the basis $\{1, X, X^2, \dots, X^n, \dots\}$, and $R[X]$ is an R-algebra. (Exercise 4).

__Lemma__ 1.5.3 If A and B are R-algebras, then $A \otimes_R B$ has a natural structure of R-algebra.

__Proof__ As we have seen in Proposition 1.4.2 the abelian group $A \otimes_R B$ has a natural structure of R-module, given by

$$r(a \otimes b) = (ra) \otimes b = a \otimes (rb)$$

It is routine to check that $A \otimes_R B$ is actually an R-algebra. q.e.d.

We say that the R-algebra A is __graded by (or over) G__, where $(G, \cdot)$ is a group, if there is a family of R-submodules $(A_g)_{g\in G}$ of A such that

$$A = \bigoplus_{g\in G} A_g$$

$$A_g A_h = A_{gh} \qquad \text{for all } g, h \in G$$

where $A_g A_h = \{a_g a_h \mid a_g \in A_g,\ a_h \in A_h\}$. Elements in A_g $(g \in G)$ are called __homogeneous__ of degree g. Observe that the condition $A = \bigoplus_{g\in G} A_g$ means that every element $a \in A$ has a unique decomposition as a finite sum of homogeneous elements $a = \sum_{g\in G} a_g$ (almost all a_g are zero.) (a_g $(g \in G)$ are called homogeneous component of a) We shall say simply that A is __graded__ if it is graded over $(\mathbb{Z}, +)$ i.e. $A = \bigoplus_{n\in\mathbb{Z}} A_n$ with $A_n \cdot A_m \subseteq A_{n+m}$.

A is __positively (negatively) graded__ if it is graded over $\mathbb{Z}$ and $A_n = 0$ for all $n < 0$ (resp. $n > 0$). Note that an ordinary R-alg. A is trivially graded by putting $A_0 = A$, $A_n = 0$, $n \neq 0$.

(Mainly we work with ~~$\mathbb{Z}$-graded~~ algebras graded over $\mathbb{Z}$).

__Example__ 1.5.4. a) The group algebra $R[G]$ is graded by G. (__Exercise__)

b) The polynomial algebra $R[X]$ is graded by $\mathbb{Z}$.

If $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and $B = \bigoplus_{n\in\mathbb{Z}} B_n$ are two graded R-algebras, an R-algebra homomorphism $f : A \to B$ is called __graded__ if $f(A_n) \subseteq B_n$ for all $n \in \mathbb{Z}$. An ideal (submodule) I of A is called __graded__ if $I = \sum_{n\in\mathbb{Z}} (I \cap A_n) = \bigoplus_{n\in\mathbb{Z}} (I \cap A_n)$, or equivalently, if every $a \in A$ belongs to I exactly if all its homogeneous components are in I

<u>Proposition</u> 1.5.5. If A is a graded R-algebra and I is a graded ideal of A, then the quotient ring has a decomposition $A/I = \bigoplus_{n\in\mathbb{Z}} (A_n+I)/I$ making it into a graded R-algebra, such that the canonical projection $p_I : A \longrightarrow A/I$ is graded.

<u>Proof</u>. <u>Exercise 7</u>

Let M be an R-module (recall that R is commutative). For all $n\in\mathbb{Z}$, $n\geq 0$, we define

$$M^{\otimes n} = M\otimes_R M\otimes_R \dots \otimes_R M \qquad (n\text{-factors})$$

Clearly $M^{\otimes 0} = R$, $M^{\otimes 1} = M$, $M^{\otimes 2} = M\otimes_R M$ and so on. By Exercise 8, Section 1.4, we know for all $n, m\geq 0$ that $M^{\otimes n}\otimes_R M^{\otimes m} \cong M^{\otimes(n+m)}$. The canonical bilinear map is given by

$$M^{\otimes n}\times M^{\otimes m} \longrightarrow M^{\otimes n}\otimes_R M^{\otimes m} = M^{\otimes(n+m)}, \quad (z,t)\mapsto z\otimes t$$

All these maps for $n,m\geq 0$ define a multiplication on the R-module

$$T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$$

making it into a graded R-algebra, called the <u>tensor algebra</u> of $_RM$. More precisely, if

$$z = \sum_{n\geq 0} z_n, \quad t \in \sum_{n\geq 0} t_n \in T(M) \qquad (\text{that is almost all } z_n \text{ are } 0$$

and almost all t_n are 0) we define

$$z\cdot t = \sum_{n\geq 0}\Big(\sum_{i+j=n} z_i\otimes t_j\Big).$$

Since every homogeneous element in $T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ is a finite sum of tensor monomial (i.e. expression of the form $x_1\otimes\dots\otimes x_n$) and every element in $T(M)$ is a finite sum of homogeneous elements, and the multiplication in $T(M)$ is distributive w.r.t. the addition, it ~~suffices to~~ is sufficient to say how ~~does~~ do multiply in $T(M)$, them tensor monomial. ~~Thus~~ If $x_1\otimes\dots\otimes x_n\in M^{\otimes n}$, $y_1\otimes\dots\otimes y_m\in M^{\otimes m}$ we obtain

$$(x_1\otimes\dots\otimes x_n,\ y_1\otimes\dots\otimes y_m)\mapsto x_1\otimes x_2\otimes\dots\otimes x_n\otimes y_1\otimes\dots\otimes y_m\in M^{\otimes(n+m)}$$

Denote by $\xi_M : M = M^{\otimes 1}\longrightarrow \bigoplus_{n\geq 0} M^{\otimes n} = T(M)$ the canonical injection into the direct sum, and call it the <u>canonical homomorphism</u> of the tensor algebra.

Theorem 1.5.6 (The universal property of the tensor algebra). Let M be an R-module, where R is a commutative ring and let $\xi_M : M \to T(M)$ the canonical homomorphism of the tensor algebra. Then for every R-algebra A and every R-homomorphism $f : M \to A$, there is a unique R-algebra homomorphism $\bar{f} : T(M) \to A$ such that $f = \bar{f} \cdot \xi_M$. Moreover this universal property determines the tensor algebra, up to a unique isomorphism.

Proof Consider the R-linear maps ~~$f: M$~~ $f^n : M^{\otimes n} \to A$, $n \geq 0$ constructed as follows: $f^0 : R = M^{\otimes 0} \to A$ is given by $f^0(r) = r \cdot 1$ (see Lemma 1.5.1); $f^1 : M = M^{\otimes 1} \to A$, $f^1 = f$; for $n \geq 2$ the map

$$\cancel{M \otimes_R M \otimes} \qquad M^n = M \times M \times \dots \times M \to A, \quad (x_1, x_2, \dots, x_n) \mapsto f(x_1) f(x_2) \dots f(x_n)$$

is multilinear, so it induces a unique R-linear map

$$f^n : M \otimes_R M \otimes_R \dots \otimes_R M = M^{\otimes n} \to A.$$

($f^n(x_1 \otimes \dots \otimes x_n) = f(x_1) f(x_2) \dots f(x_n)$) see also Exercise 9, Section 1.4.

By the universal property of the direct sum there is a unique R-linear map $\bar{f} : T(M) = \bigoplus_{n \geq 0} M^{\otimes n} \to A$ making commutative the diagram

$$\begin{tikzcd} M^{\otimes n} \arrow[r, "\xi_n"] \arrow[dr, "f^n"'] & T(M) \arrow[d, "\bar{f}"] \\ & A \end{tikzcd}$$

for all $n \geq 0$, where by $\xi_n : M^{\otimes n} \to \bigoplus_{n \geq 0} (M^{\otimes n}) = T(M)$, we denoted the canonical injections of the direct sum. ~~For $n = 1$~~ The commutativity of the diagram for $n = 1$ means $\bar{f} \cdot \xi_1 = \bar{f} \cdot \xi_M = f$.

~~If $\tilde{f} : T(M) \to A$ is another R-linear map such that $\tilde{f} \cdot \xi_M = f$~~

In order to show that $\bar{f}$ is a homomorphism of R-algebra, it is sufficient to show that $\bar{f}$ commutes with the product of two tensor monomials. Let $x = x_1 \otimes \dots \otimes x_n \in M^{\otimes n}$ and $y = y_1 \otimes \dots \otimes y_m \in M^{\otimes m}$. Then $\bar{f}(x \otimes y) = \bar{f}(x_1 \otimes \dots \otimes x_n \otimes y_1 \otimes \dots \otimes y_m) = f(x_1) \dots f(x_n) f(y_1) \dots f(y_m) =$
$= \bar{f}(x) \bar{f}(y)$.

If $\tilde{f}: T(M) \to A$ is another R-algebra homomorphism such that $\tilde{f}\cdot\xi_M = f$, then since every tensor monomial $x = x_1 \otimes \dots \otimes x_n \in M^{\otimes n}$ is a product in $T(M)$ of $x_1, x_2, \dots, x_n \in M^{\otimes 1} = M$, we have ~~$\tilde{f}(x) = \tilde{f}(x_1 \otimes \dots \otimes x_n) =$~~

~~$= \tilde{f}(x_1) \dots \tilde{f}(x_n) = (\tilde{f}\cdot\xi_M)(x_1)$~~

$$\tilde{f}(x) = \tilde{f}(x_1 \otimes \dots \otimes x_n) = (\tilde{f}\cdot\xi_M)(x_1) \dots (\tilde{f}\cdot\xi_M)(x_n) = f(x_1)\dots f(x_n) = \bar{f}(x)$$

~~Thus $\tilde{f}$ and~~ Thus $\tilde{f}$ and $\bar{f}$ coincide on tensor monomials so $\tilde{f} = \bar{f}$. The last statement concerning the fact that $T(M)$ is determined up to an isomorphism by its universal property is Exercise 8.

Corollary 1.5.7 For every two R-modules M and N and every R-linear map $f: M \to N$, there is a unique homomorphism of R-algebras $T(f): T(M) \to T(N)$ making commutative the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\xi_M"'] & N \arrow[d, "\xi_N"] \\ T(M) \arrow[r, dashed, "T(f)"] & T(N) \end{tikzcd}$$

where $\xi_M: M \to T(M)$ and $\xi_N: N \to T(N)$ are the canonical homomorphisms.

Lemma 1.5.8 Let A be an R-algebra and $I(A)$ be the ideal generated by all elements of the form $xy - yx$, with $x, y \in A$. That is

$$I(A) = \langle \{ xy - yx \mid x, y \in A \} \rangle \qquad \text{(see Exercise 21, section 1.1)}$$

Denote by $\lambda_A: A \to A/I(A)$ the canonical projection. Then $A/I(A)$ is a commutative R-algebra (in the sense that the ring $(A/I(A), +, \cdot)$ is commutative) and for every commutative R-algebra B and every homomorphism of R-algebras $f: A \to B$, there is a unique homomorph. of R-algebras $f': A/I(A) \to B$ s.t. $f' \lambda_A = f$.

Proof. $A/I(A) = \{ \bar{x} \mid x \in A \}$ where $\bar{x} = x + I(A)$. For $x, y \in A$ we have $xy - yx \in I(A)$ so $\bar{0} = \overline{xy - yx} = \bar{x}\bar{y} - \bar{y}\bar{x}$, thus $\bar{x}\cdot\bar{y} = \bar{y}\cdot\bar{x}$ in $A/I(A)$. Moreover $f(xy - yx) = f(x)\cdot f(y) - f(y)f(x) = 0$ in B, since B is commutative. Therefore $I(A) \subseteq \operatorname{Ker} f$ and the theorem of factorization through a surjection (see Exercise 14, section 1.1) gives a unique R-linear map

$f': A/I(A) \longrightarrow B$ such that $f' \lambda_A = f$ (see the diagram above).

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A"'] & B \\ A/I(A) \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

For $\bar{x}, \bar{y} \in A/I(A)$ we have

$$f'(\bar{x}\cdot\bar{y}) = f'(\overline{xy}) = f(xy) = f(x)f(y) = f'(\bar{x})\cdot f'(\bar{y})$$

so f' is a homomorphism of R-algebras q.e.d.

Consider now an R-module M and $I(T(M)) = \langle \{xy - yx \mid x, y \in T(M)\} \rangle$ the ideal of the tensor algebra $T(M)$ generated by all $xy - yx$, $x, y \in M$. The commutative algebra $Sym(M) = T(M)/I(T(M))$ is called the <u>symmetric algebra</u> of M. Denote by $\eta_M : M \longrightarrow Sym(M)$ the composition between $\xi_M : M \longrightarrow T(M)$ and the canonical projection $\lambda_{T(M)} : T(M) \longrightarrow T(M)/I(T(M))$, and call η_M the canonical homomorph. of the symm. algebra.

~~Theorem 1.5.9 Let M be an R-module, where R is a commutative ring.~~

<u>Theorem</u> 1.5.9 (The universal property of the symmetric algebra). Let M be an R-module, where R is a commutative ring. For every commutative R-algebra A and every R-linear map $f: M \longrightarrow A$, there is a unique homomorphism of R-algebras $f': Sym(M) \longrightarrow A$ such that $f' \eta_E = f$. Moreover this universal property determines the symmetric algebra up to a unique isomorphism of R-algebras.

<u>Proof</u>. We have a comm. diagram of the form

$$\begin{tikzcd} M \arrow[rr, "\eta_M"] \arrow[dr, "\xi_M"] \arrow[ddr, "f"'] & & Sym(M) \arrow[ddl, dashed, "\exists! f'"] \\ & T(M) \arrow[ur, "\lambda_{T(M)}"] \arrow[d, "\exists! \bar{f}"] & \\ & A & \end{tikzcd}$$

~~The homomorphism of R-algebras $\bar{f}: T(M) \to A$~~

The existence and unicity of the homomorphism of R-algebras $\bar{f}$ follow from the universal property of the tensor algebra (Theorem 1.5.6) and the existence and unicity of f' follow by Lemma 1.5.8. The fact that $Sym(M)$ is determined up to a unique isomorphism by its universal property is Exercise 9.

Again M is an R-module, when R is a commutative ring. Let denote by

$$J(M) = \langle \{ x \otimes x \mid x \in M, x \neq 0 \} \rangle$$

the ideal of $T(M)$ generated by all homogeneous elements of degree 2 of the form $x \otimes x$, $x \in M$. ~~Being generated by~~

<u>Lemma</u> 1.5.10 If $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a graded R-algebra and I is an ideal in A generated by homogeneous elements, then I is graded.

<u>Proof</u> <u>Exercise 10</u>.

We return to the ideal $J(M)$ of $T(M)$. By Lemma 1.5.10 it is graded, so

$$J(M) = \bigoplus_{n \geq 0} \left(J(M) \cap M^{\otimes n} \right)$$

The graded algebra

$$\Lambda(M) = T(M)/J(M) = \bigoplus_{n \geq 0} \left(M^{\otimes n} + J(M) \right) / J(M)$$

is called the <u>exterior algebra</u> of M. We denote by

$$M^{\wedge n} = \left(M^{\otimes n} + J(M) \right) / J(M) \cong M^{\otimes n} / M^{\otimes n} \cap J(M)$$

an we call ~~this R-mo~~ it the <u>n-th exterior power</u> of M. Clearly $M^{\wedge n}$ is an R-module and

$$\Lambda(M) = \bigoplus_{n \geq 0} M^{\wedge n}.$$

Since $J(M)$ is generated by homogeneous elements of degree 2, we deduce $M^{\wedge 0} = R \cap J(M) = R$, $M^{\wedge 1} = M \cap J(M) = M$. If $z, t \in \Lambda(M)$ we denote by $z \wedge t$ the product in $\Lambda(M)$. This product is called the <u>exterior product</u> of z and t.

<u>Remark</u> 1.5.11 a) For every $x \in M$ we have $x \wedge x = 0$. <u>Exercise 11</u>.

b) For every $x, y \in M$ we have $x \wedge y = -y \wedge x$. <u>Exercise 12</u>.

Note that the elements of the form $x_1 \wedge x_2 \wedge \dots \wedge x_n$, $x_i \in M$, $1 \leq i \leq n$ form a family of generators of $\Lambda(M)$, since the tensor monomials $x_1 \otimes \dots \otimes x_n$ generate $T(M)$ as R-module. We denote by $\varphi_M : M \to \Lambda(M)$ the composition between $\xi_M : M \to T(M)$ and the canonical projection $T(M) \to T(M)/J(M) = \Lambda(M)$. Then φ_M is a R-linear map called the <u>canonical homomorphism</u> of the exterior algebra.

Theorem 1.5.12 (The universal property of the exterior algebra). Let M be an R-module, where R is a commutative ring. For every R-algebra A and every R-linear map $f: M \to A$ such that $f(x)^2 = 0$, for all $x \in M$, there is a unique homomorphism of R-algebras $f^{\#}: \Lambda(M) \to A$ such that $f^{\#} \cdot \psi_M = f$.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "\psi_M"] \arrow[r, "\xi_M"] \arrow[dr, "f"'] & T(M) \arrow[r] \arrow[d, dashed, "\exists! \bar{f}"] & \Lambda(M) \arrow[dl, dashed, "\exists! f^{\#}"] \\
 & A &
\end{tikzcd}$$

The existence and the unicity of $\bar{f}$ follows by the universal property of $T(M)$ (Theorem 1.5.6). Since $f(x)^2 = 0$ for all $x \in M$, we obtain

$\bar{f}(x \otimes x) = f(x)^2 = 0$ so $J(M) \subseteq \operatorname{Ker} \bar{f}$.

Thus the existence and the unicity of the R-linear map $f^{\#}$ follows by the theorem of factorization through projection (Exercise 14, section 1.1). Now we have to check that $f^{\#}$ is a homomorphism of R-algebras. It is sufficient to verify that $f^{\#}$ commutes with the product of generators of $\Lambda(M)$ i.e. with products of elements of the form $x_1 \wedge \dots \wedge x_n$, $x_1, \dots, x_n \in M$. But

$$\begin{aligned}
f^{\#}(x_1 \wedge \dots \wedge x_n, y_1 \wedge \dots \wedge y_m) &= \bar{f}(x_1 \otimes \dots \otimes x_n, y_1 \otimes \dots \otimes y_m) = \cancel{x_1 \otimes \dots \otimes x_n \otimes y_1 \otimes \dots \otimes y_m} \\
&= f(x_1) \dots f(x_n) f(y_1) \dots f(y_m) = \bar{f}(x_1 \otimes \dots \otimes x_n)\, \bar{f}(y_1 \otimes \dots \otimes y_m) = \\
&= f^{\#}(x_1 \wedge \dots \wedge x_n)\, f^{\#}(y_1 \wedge \dots \wedge y_m).
\end{aligned}$$

Exercise 13. Show that the universal property determines $\Lambda(M)$ up to a unique isomorphism i.e. if E is another R-algebra and $\theta: M \to E$ is an R-linear map such that $\theta(x)^2 = 0$ for all $x \in M$ satisfying the same universal property as $\Lambda(M)$ then there is a unique isomorphism of R-algebras $\varphi: \Lambda(M) \to E$ such that $\varphi \cdot \psi_M = \theta$.

Corollary 1.5.13 If M and N are two modules over a commutative ring R, and $f: M \to N$ is an R-linear map, then there is a unique R-algebra homomorphism $\Lambda(f): \Lambda(M) \to \Lambda(N)$ making commutative the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\psi_M"'] & N \arrow[d, "\psi_N"] \\
\Lambda(M) \arrow[r, "\Lambda(f)"] & \Lambda(N)
\end{tikzcd}$$

Let K be a field. A Lie algebra $\underline{g}$ over K is a vector space over K ($=$ K-module) together with a bilinear map

$$[-,-] : \underline{g} \times \underline{g} \longrightarrow \underline{g}$$

called the Lie bracket, satisfying the following two axioms:

$$[x,x]=0, \text{ for all } x \in \underline{g};$$

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \text{ for all } x,y,z \in \underline{g}$$

The last ~~inequality~~ equality is called the Jacobi identity.

Exercise 14 Show that, in every Lie algebra $\underline{g}$, it holds

$$[x,y] = -[y,x] \qquad \text{for all } x,y \in \underline{g}.$$

A Lie algebra homomorphism is a K-linear map $f : \underline{g} \longrightarrow \underline{h}$ with $f[x,y] = [f(x), f(y)]$ for all $x,y \in \underline{g}$. A Lie subalgebra $\underline{h}$ of $\underline{g}$ is a K-subspace of $\underline{g}$ closed under $[-,-]$. A Lie subalgebra $\underline{h}$ is called a Lie ideal if $[x,y] \in \underline{h}$ for all $x \in \underline{g}$ and all $y \in \underline{h}$. If $\underline{h}$ is a Lie ideal of $\underline{g}$ then the quotient space $\underline{g}/\underline{h}$ has a natural Lie algebra structure such that the canonical projection $p_{\underline{h}} : \underline{g} \longrightarrow \underline{g}/\underline{h}$ is a Lie algebra homomorphism Exercise 15.

A Lie algebra $\underline{g}$ is called abelian if $[x,y]=0$ for all $x,y \in \underline{g}$. Clearly any K-vector space may be regarded as an abelian Lie algebra, by defining $[x,y]=0$ for all ~~$x,y \in$~~ vectors x,y. If $\underline{g}$ is a Lie algebra then the K-subspace $[\underline{g},\underline{g}]$ of $\underline{g}$ generated by all ~~above~~ elements of the form $[x,y]$ with $x,y \in \underline{g}$ is an ideal ~~Exercise~~ and the quotient $\underline{g}/[\underline{g},\underline{g}]$ is the largest abelian quotient of $\underline{g}$ (Exercise 16). Given any K-algebra A we can associate a Lie algebra structure on A by defining:

$$[-,-] : A \times A \longrightarrow A, \quad [x,y] = xy - yx$$

(Verify that we obtain a Lie algebra structure in this way Exercise 17)
We denote also by A this Lie algebra.

Conversely being given a Lie algebra $\underline{g}$ we construct a K-algebra called the <u>universal enveloping algebra</u> of g by setting

$$\mathcal{U}(\underline{g}) = T(\underline{g}) / \langle \{ x\otimes y - y\otimes x - [x,y] \mid x,y\in \underline{g} \} \rangle$$

where $T(\underline{g})$ is the tensor algebra of the K-module $\underline{g}$ and $\langle \{ x\otimes y - y\otimes x - [x,y] / x,y\in \underline{g} \} \rangle$ is the ideal of $T(\underline{g})$ generated by the indicated elements. Denote by $\iota_{\underline{g}} : \underline{g} \longrightarrow \mathcal{U}(\underline{g})$ the R-linear map obtained by composing $\xi_{\underline{g}} : \underline{g} \longrightarrow T(\underline{g})$ with the canonical projection $T(\underline{g}) \longrightarrow \mathcal{U}(\underline{g})$, and call it the <u>canonical homomorphism of the universal enveloping algebra</u>.

<u>Theorem</u> 1.5.14 (The universal property of the universal enveloping alg.) If $\underline{g}$ is a Lie algebra over a field K and A is a K-algebra, then for every Lie algebra homomorphism $f : \underline{g} \longrightarrow A$ there is a unique K-algebra homomorphism $\tilde{f} : \mathcal{U}(\underline{g}) \longrightarrow A$ such that $\tilde{f}\circ\iota_{\underline{g}} = f$.

$$\begin{tikzcd} \underline{g} \arrow[r, "\iota_{\underline{g}}"] \arrow[d, "f"'] & \mathcal{U}(\underline{g}) \arrow[dl, dashed, "\tilde{f}"] \\ A & \end{tikzcd}$$

Moreover this universal property determines $\mathcal{U}(\underline{g})$ up to a unique isomorphism of K-algebras.

<u>Proof</u> (Exercise 18)

<u>Corollary</u> 1.5.15 If $\underline{g}$ and $\underline{h}$ are Lie algebras over K, ~~then~~ for every Lie algebra homomorphism $f : \underline{g} \longrightarrow \underline{h}$ there is a unique K-algebra homomorphism $\mathcal{U}(f) : \mathcal{U}(\underline{g}) \longrightarrow \mathcal{U}(\underline{h})$ making commutative the diagram

$$\begin{tikzcd} \underline{g} \arrow[r, "f"] \arrow[d, "\iota_{\underline{g}}"'] & \underline{h} \arrow[d, "\iota_{\underline{h}}"] \\ \mathcal{U}(\underline{g}) \arrow[r, dashed, "\mathcal{U}(f)"] & \mathcal{U}(\underline{h}) \end{tikzcd}$$

<u>Exercise 19</u> If A and B are R-algebras then

$$A \otimes_R B[X] \cong A[X] \quad , \qquad A \otimes_R M_n(B) \cong M_n(A) .$$