INTRODUCTION TO HOMOLOGICAL ALGEBRA

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Contents

Chapter 0

Course description

0.1 Topics of the course

The following topics will be discussed in this course.
- triangulated spaces and simplicial sets;
- chain complexes and homology;
- modules and algebras;
- categories, functors and natural transformations;
- derived functors.

0.2 Location

Class meets on Wednesdays 14:00 – 15:50 in Mathematicum Lecture Room e.

0.3 Grading

The students will get the grades according to the following rules.

1. The students will get points (from 0.5 to 2 or 3) for homeworks (exercises given during the course).
2. The points for an exercise or a program will be awarded to only one student, to avoid copying.
3. 10 points are equal with the grade 10 (and so one).

Chapter 1

Simplicial Sets

1.1 Triangulated Spaces

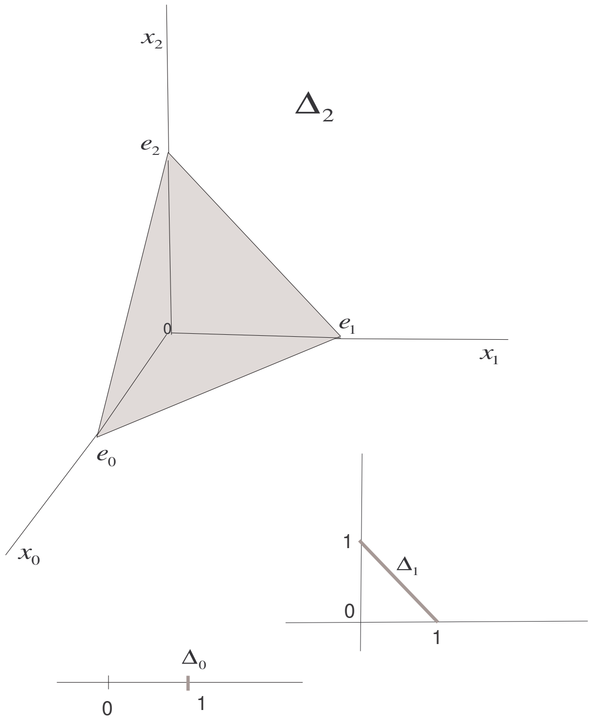
These spaces can be glued from simplices (points, segments, triangles, tetrahedrons,... higher dimensional simplices). They can be described combinatorially, by giving:

(1) the number of simplices of any dimension;

(2) how they are glued together.

Definition 1.1.1 The n-dimensional **simplex** is the topological space

$$\Delta_n = \{(x_0, \ldots x_n) \in R^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\ \forall i = 0, \ldots, n\}$$



The point $e_i = (0, \ldots, \underset{i}{1}, \ldots, 0)$ is the i-th vertex of Δ_n. The vertices are ordered as $e_0 < e_1 < \cdots < e_n$. Denote $[n] = \{0, 1, 2, \ldots n\}$. If $I \subset [n]$, then the I-th face of Δ_n is

$$\{(x_0, \ldots x_n) \in \Delta_n \mid x_i = 0\ \forall i \notin I\}$$

Instead of a subset $I \subset [n]$ we may take $0 \leq m \leq n$ and the increasing map $f : [m] \to [n]$ such that $\mathrm{Im} f = I$, where $|I| = m + 1$. It is clear that there exists a unique linear map $\Delta_f : \Delta_m \to \Delta_n$ that preserves the order of vertices and has the I-th face as image.

Example 1.1.2 a) If $I = \{1, 2\} \subset [2]$, then the I-th face of Δ_2 is the segment $e_0 e_1$.

b) Let $f:[1]\to[2]$ be defined by $f(0)=0$, $f(1)=2$, so Imf $=I=\{0,2\}\subset[2]$. Then $\Delta_f:R^2\to R^2$, $\Delta_f(1,0)=(1,0,0)$, $\Delta_f(0,1)=(0,0,1)$, and the matrix of Δ_f is

$$[\Delta_f]=\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}.$$

Then $\Delta_f:\Delta_1\to\Delta_2$ has as image the segment e_0e_1.

Definition 1.1.3 A **gluing data** is the following set $X_\bullet$ of structures:

a) *What we glue*: $X_{(0)}$- points, $X_{(1)}$-segments, $X_{(2)}$-triangles, $\dots X_{(n)}$-n-dimensional simplices.

b) *How to glue*: for any pair $\{n:I\subset[n]\}$, $|I|=m+1$ a map $:X_{(n)}\to X_{(m)}$ is given that specifies which m-dimensional simplex should be identified with the I-th face of the corresponding n-dimensional simplex.

More precisely, let a face correspond to an increasing map $f:[m]\to[n]$. Let $X(f):X_{(n)}\to X_{(m)}$ be the corresponding gluing map. The family $\{X(f)\}_f$ should satisfy the following conditions:

$$X(\mathrm{id})=\mathrm{id},\quad X(g\circ f)=X(g)\circ X(f).$$

This means that two different elements of $X_{(m)}$ correspond to different simplices and that a face of a simplex ...

Definition 1.1.4 For a simplicial set X, let $|X|$ be the topological space with underlying set

$$|X|=\bigcup_{n=0}^{\infty}(\Delta_n\times X_{(n)})/\rho,$$

where ρ is the smallest equivalence relation that satisfies

$$(x,s)\rho(t,y)\ \ \forall(s,x)\in\Delta_n\times X_{(n)},(t,y)\in\Delta_m\times X_{(m)}:y=X(f)x,\quad s=\Delta_f(t)$$

for some increasing map $f:[m]\to[n]$. The canonical topology on $|X|$ is the weakest topology such that the canonical projection $\tau:\bigcup_{n=0}^{\infty}(\Delta_n\times X_{(n)})\to|X|$ is continuous.

c) The space $|X|$ with the gluing data $X_\bullet$ is called a **triangulated space**.

For instance, in the case of the 3-dimensional simplex, we have $|X_{(0)}|=4$, $|X_{(1)}|=6$, $|X_{(2)}|=4$, $|X_{(3)}|=1$. Informally, we say that:

- Each point of $|X|$ belongs to at least one simplex;
- Each point of $|X|$ belongs to finitely many simplices;
- The set of points of two simplices is either the empty set or is a face or a face of a face ... of each simplex.

Example 1.1.5 Examples of triangulated spaces:

a) The n-dimensional simplex Δ_n with the standard triangulation:

$$X_{(i)}=\text{ subsets of cardinality } i+1 \text{ in } [n]=\text{ increasing maps from } [i] \text{ to } [n].$$

If $f:[i]\to[j]$ is an increasing map then $X(f)$ maps the simplex $g:[j]\to[n]$ into $g\circ f:[i]\to[n]$.

b) The *sphere* S^n is obtained from the standard triangulation of Δ_{n+1} by deleting the unique $n+1$-dimensional simplex.

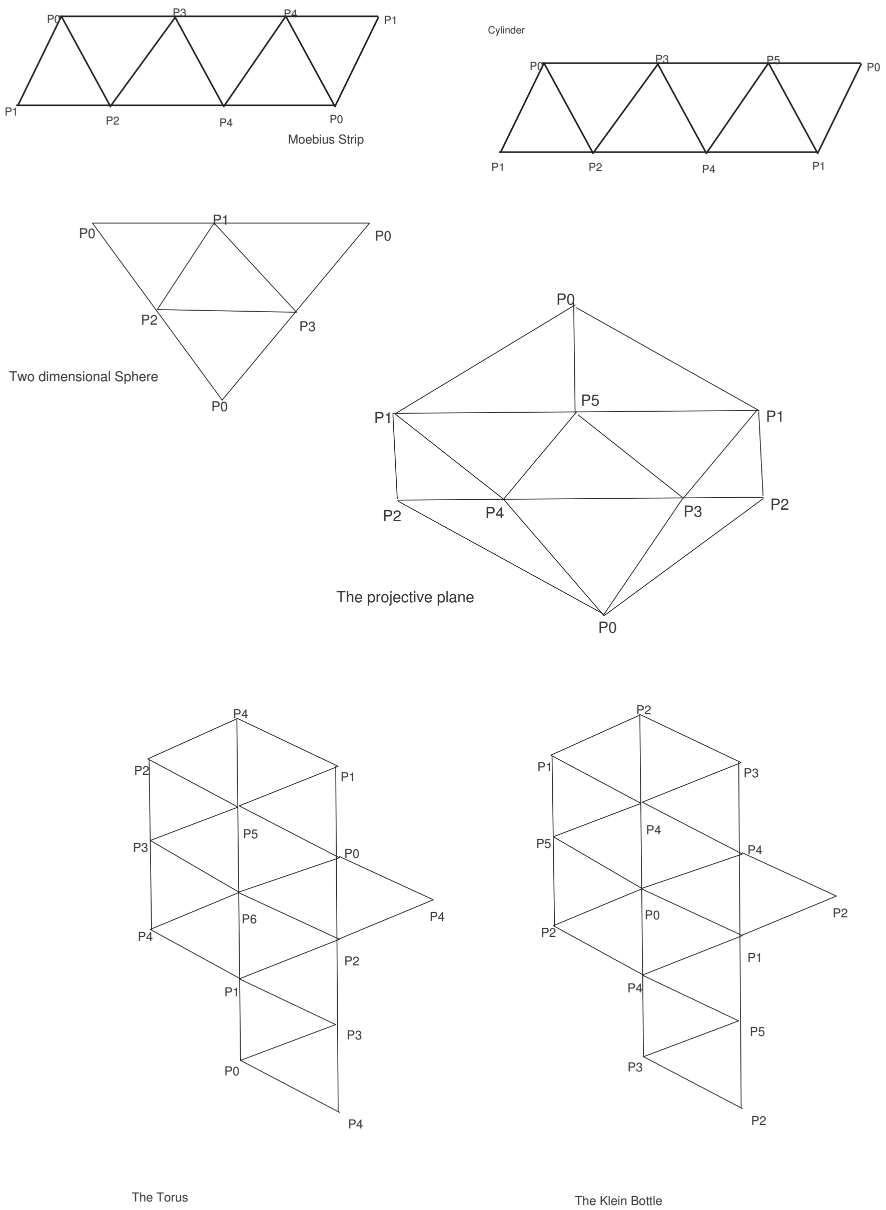
Exercise 1 Verify that Δ_n is indeed a triangulated space.

Proposition 1.1.6 *Any triangulated space is the set theoretical disjoint union of the interior of its simplices. Namely, let*

$$\overset{\circ}{\Delta}_n=\begin{cases}\mathrm{int}(\Delta_n), & n\geq 1\\ \Delta_0, & n=0\end{cases},\quad \mathrm{int}(\Delta_n)=\{(x_0,\dots x_n)\in R^{n+1}\mid\sum_{i=0}^{n}x_i=1,x_i>0,\ i\in[n]\},\ n\geq 1.$$

Let $(X_{(i)},X(f))$ *be some gluing data. Let* $\tau:\bigcup_{n=0}^{\infty}(\Delta_n\times X_{(n)})\to|X|$ *be the (continuous) triangulation map. Then* τ *induces the bijective map*

$$\overset{\circ}{\tau}:\bigcup_{n=0}^{\infty}(\overset{\circ}{\Delta}_n\times X_{(n)})\to|X|$$



Triangulations of some topological spaces

1.2 Simplicial Sets

Definition 1.2.1 A **simplicial set** X is a family of sets $X_\bullet = (X_n)_{n\geq 0}$ and maps $X(f) : X_n \to X_m$ for each nondecreasing map $f : [m] \to [n]$, such that

$$X(\mathrm{id}) = \mathrm{id}, \quad X(g \circ f) = X(g) \circ X(f).$$

(The difference with the gluing data introduced in 1.1.3 is that f need not be strictly increasing.)

The elements of X_n are called **n-simplices**. Instead of $X_\bullet$ we often denote X.

For any nondecreasing map $f : [m] \to [n]$ define the f-th *face* Δ_f as the linear map from Δ_m into Δ_n that maps $e_i \in \Delta_m$ to $e_{f(i)} \in \Delta_n$, $\forall i \in [m]$. (The difference to 1.1.1 is that Δ_f need not be embedding. If f is not strictly increasing, then Δ_f decreases the dimension since we glue together some vertices of Δ_n.)

Definition 1.2.2 The geometric realization $|X|$ of the simplicial set $X_\bullet$ is the topological space with underlying set $|X| = \bigcup\limits_{n=0}^{\infty} (\Delta_n \times X_n)/\rho$ where ρ is the smallest equivalence relation that satisfies

$$(s, x)\rho(t, y) \;\forall (s, x) \in \Delta_n \times X_n,\; (t, y) \in \Delta_m \times X_m :\; y = X(f)x,\; s = \Delta_f(t)$$

for some nondecreasing map $f : [m] \to [n]$. The topology on $|X|$ is the weakest one for which the canonical map $\bigcup\limits_{n=0}^{\infty} (\Delta_n \times X_n) \to |X|$ is continuous.

Remark 1.2.3 One can associate to each triangulation a simplicial set with the same geometric realization, but the notion of simplicial set is more general.

1.3 Chains and cochains

The boundary of $\Delta_1 = (P_0P_1)$ is the difference $P_1 - P_0$ of its vertices. This is similar to the Newton-Leibnitz formula

$$\int_0^1 f'(x)dx = f(1) - f(0)$$

Similarly, the boundary of Δ_n is the alternating sum of its $n-1$ dimensional faces.

Definition 1.3.1 Let X be a simplicial set.

a) Let $C_n(X)$ be the free abelian group generated by all n-simplices of X

$$C_n(X) = \{\sum_{x\in X_n} a_x x \mid a_x \in \mathbb{Z},\ a_n \neq 0 \text{ for a finite number of simplices } x\},$$

Thus $C_n(X)$ is the free $\mathbb{Z}$-module with basis X_n. The elements of $C_n(X)$ are called n-**chains** of X.

b) Let $\partial_n^i : [n-1] \to [n]$ be the unique stricly increasing map whose image is $[n] \setminus \{i\}$.

c) Let $c = \sum_{x\in X_n} a_x x \in C_n(X)$ be a n-chain. The **boundary** of c is the $n-1$-chain

$$d_n c = \sum_{x\in X_n} a_x \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x)$$

So d_n is $\mathbb{Z}$-linear, i.e. morphism of abelian groups, and for all $x \in X_n$

$$d_n x = \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x).$$

This defines the **boundary operator** $d_n : C_n(X) \to C_{n-1}(X)$, which is a $\mathbb{Z}$-linear map. For $n = 0$ we let $d_n = 0$, and $C_n(X) = \{0\}$ for $n < 0$.

d) We can generalize the definition to obtain chains with coefficients in an abelian group $(A, +)$.

$$C_n(A, X) = \{\sum_{x\in X_n} a_x x : a_x \in A\}.$$

This means that $C_n(A, X) = C_n(X) \otimes_{\mathbb{Z}} A$. In particular, $C_n(X) = C_n(X, \mathbb{Z})$.

Definition 1.3.2 a) We define **cochains** with coefficients in an abelian group $(A, +)$.

$$C^n(A, X) = \{f \mid f : X_n \to A\}$$

is called the group of n-cochains of X. So $(C^n(A, X), +)$ is also an abelian group, where

$$f + g : X_n \to A_n, \quad (f + g)(x) = f(x) + g(x).$$

b) The **coboundary operator** is the group homomorphism $d^n : C^n(A, X) \to C^{n-1}(A, X)$ defined by

$$d^n f : X_{n+1} \to A, \quad (d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i f(X(\partial_{n+1}^i)(x))$$

Lemma 1.3.3 a) $d_{n-1} \circ d_n = 0$, $\forall n \geq 1$*;*

b) $d^{n+1} \circ d^n = 0$, $\forall n \geq 0$.

Proof. a) Consider the diagram $C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} C_{n-2}(X)$. Note first that for any $0 \leqslant j < i \leqslant n-1$ we have $\partial_n^i \circ \partial_{n-1}^j = \partial_n^i \circ \partial_{n-1}^j$. Let $x \in C_n(X)$. Then

$$\begin{aligned}
(d_{n-1} \circ d_n)(x) &= d_{n-1}(\sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x)) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} (X(\partial_{n-1}^i) \circ X(\partial_n^i))(x) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} X(\partial_{n-1}^i \circ \partial_n^i)(x).
\end{aligned}$$

Composition $\partial_n^i \circ \partial_{n-1}^j$ for different i, j yields all increasing maps defined on $[n-2]$ into $[n]$ and the map whose image does not contain i and j appear exactly twice as $\partial_n^i \circ \partial_{n-1}^j$ with sign $(-1)^{i+j}$, and the second time as $\partial_n^i \circ \partial_{n-1}^j = \partial_n^j \circ \partial_{n-1}^{i-1}$ with the opposite sign $(-1)^{i+j-1}$. Hence the sum is zero, i.e. $d_{n-1} \circ d_n = 0$. ∎

Exercise 2 Prove statement b).

1.4 Complexes and homology

The motivation for the concepts introduces below comes from the following idea: we associate to a complicated object, say a topological space, something which is easier to study – an abelian group for instance. To the simplicial set X we associate its (co)homology groups: $H^\bullet(X)$ and $H_\bullet(X)$. If $X \sim Y$ (there exists some equivalence between them) then we must have an isomorphism $H_\bullet(X) \simeq H_\bullet(Y)$ of abelian groups.

Definition 1.4.1 a) A **chain complex** is a sequence of abelian groups and homomorphisms with the property that $d_n \circ d_{n+1} = 0$.

$$C_\bullet = (\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots)$$

b) A **cochain complex** is a sequence of abelian groups and homomorphisms with the property that $d^n \circ d^{n-1} = 0$.

$$C^\bullet = (\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n-1} \to \cdots)$$

Remark 1.4.2 Any chain complex can be transformed into a cochain complex by letting $D^n := C_{-n}, d^n = d_{-n-1}$.

Definition 1.4.3 Let $C_\bullet$ be a chain complex and $C^\bullet$ a cochain complex.

a) The group of n-**cycles** is

$$Z_n(C_\bullet) = \text{Ker}\, d_n = \{x \in C_n \mid d_n = 0\} \leqslant C_n.$$

The group of n-**boundaries** is

$$B_n(C_\bullet) = \text{Im}\, d_{n+1} = \{d_{n+1} \mid y \in C_{n+1}\} \leqslant C_n.$$

Note that $B_n(C_\bullet) \leqslant Z_n(C_\bullet)$.

b) The group of n-**cocycles** is

$$Z^n(C^\bullet) = \text{Ker}\, d^n = \{x \in C^n \mid d^n = 0\} \leqslant C^n$$

The group of n-**coboundaries** is

$$B^n(C^\bullet) = \text{Im}\, d^{n-1} = \{d^{n-1}(y) \mid y \in C^{n-1}\} \leqslant C^n.$$

c) If $c, c' \in C_n$, we say that c, c' are **homologous** ($c \sim c'$) if

$$c - c' \in B_n(C_\bullet) \Leftrightarrow \exists c'' \in C^{n+1} : \ c - c' = d_n(c'')$$

Similarly, if $f, f' \in C^m$ are cohomologous ($f \sim f'$) if

$$f - f' \in B^n(C^\bullet) \Leftrightarrow \exists f'' \in C^n : f - f' = d^{n-1}(f'')$$

d) The **homology groups** of the chain complex $C_\bullet$ are the factor groups

$$H_n(C_\bullet) = \frac{\ker d_n}{\text{Im}\, d_{n+1}} = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)}$$

The **cohomology groups** of the cochain complex $C^\bullet$ are factor groups

$$H^n(C^\bullet) = \frac{\ker d^n}{\text{Im}\, d^{n-1}} = \frac{Z^n(C^\bullet)}{B^n(C^\bullet)}$$

If X is a simplicial set, we denote $H_n(X, A) := H_n(C_\bullet(X, A))$ (the n-th homology group of X with coefficients in $(A, +)$), and $H^n(X, A) := H^n(C^\bullet(X, A))$ (the n-th cohomology group of X with coefficients in $(A, +)$).

Recall that if B is a subgroup of $(A, +)$, then

$$A/B = \{a + B \mid a \in A\} = A/\sim,$$

where $\sim$ is defined by $a \sim a' \Leftrightarrow a - a' \in B$.

Example 1.4.4 We calculate the homology of the sphere S^2, space which is homeomorphic to the surface of the tetrahedron $P_0P_1P_2P_3$. The groups of chains are

$$C_0(S^2) = \langle P_0, P_1, P_2, P_3\rangle = \{n_0P_0 + n_1P_1 + n_2P_2 + n_3P_3 \mid n_i \in \mathbb{Z}, i = 0, \ldots, 3\},$$

$$C_1(S^2) = \langle P_0P_1, P_0P_2, P_0P_3, P_1P_2, P_1P_3, P_2P_3\rangle\,,$$

$$C_2(S^2) = \langle P_0P_1P_2, P_0P_1P_3, P_0P_2P_3, P_1P_2P_3\rangle$$

$$C_n(S^2) = 0, \quad \forall n \in \mathbb{Z}\backslash\{0, 1, 2\}.$$

Consider the complex

$$C_\bullet(S^2) = (\cdots \to 0 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0 \to \ldots).$$

Let $n = 0$. We have

$$Z_0 = \operatorname{Ker} d_0 = \{x \in C_0 \mid d_0(x) = 0\} = C_0$$

$$\begin{aligned} B_0 &= \operatorname{Im} d_1 = \langle d_1(P_0P_1), d_1(P_0P_2), \ldots, d_1(P_2P_3)\rangle \\ &= \langle P_1 - P_0, P_2 - P_0, P_3 - P_0, P_2 - P_1, P_3 - P_1, P_3 - P_2\rangle \end{aligned}$$

It follows that $\{P_1 - P_0, P_2 - P_0, P_3 - P_0\}$ is a $\mathbb{Z}$ -basis of B_0. Hence $B_0 \simeq \mathbb{Z}^3$. We want to show that $H_0 \simeq (\mathbb{Z}, +)$. We use the first isomorphism theorem. The map

$$\varphi : Z_0 \to \mathbb{Z}, \quad \varphi(n_0P_0 + n_1P_1 + n_2P_2 + n_3P_3) = n_0 + n_1 + n_2 + n_3$$

is $\mathbb{Z}$-linear, surjective, and $B_0 \subseteq \operatorname{Ker}\varphi$. Conversely, let $x = n_0P_0 + n_1P_1 + n_2P_2 + n_3P_3$ belong to $\operatorname{Ker}\varphi$. This means $\varphi(x) = 0$ and it follows immediately that $x \in B_0$. Hence $B_0 = \operatorname{Ker}\varphi$, and this implies the statement.

Let $n = 1$. By definition $H_1 = Z_1/B_1 = \operatorname{Ker} d_1/\operatorname{Im} d_2$. Let

$$x = n_1P_0P_1 + n_2P_0P_2 + n_3P_0P_3 + n_4P_1P_2 + n_5P_1P_3 + n_6P_2P \in C_1.$$

Then one easily calculates that

$$x \in Z_1 \Leftrightarrow d_1(x) = 0 \Leftrightarrow \begin{cases} n_3 &= -n_1 - n_2 \\ n_5 &= n_1 - n_4 \\ n_6 &= n_2 + n_4, \end{cases}$$

where $n_1, n_2, n_4 \in \mathbb{Z}$ are independent parameters. Then x is a 1-cycle if and only if

$$\begin{aligned} x &= n_1(P_0P_1 - P_0P_3 + P_1P_3) + n_2(P_0P_2 - P_0P_3 + P_2P_3) + n_4(P_1P_2 - P_1P_3 + P_2P_3) \\ &= n_1d_2(P_0P_1P_3) + n_2d_2(P_0P_2P_3) + n_4d_2(P_1P_2P_3) \\ &= d_2(n_1P_0P_1P_3 + n_2P_0P_2P_3 + n_4P_1P_2P_3) \end{aligned}$$

It follows that every 1-cycle is an 1-boundary, so $Z_1 \subseteq B_1$, hence $B_1 = Z_1$. Consequently, $H_1 = Z_1/B_1 \simeq \{0\}$.

Let $n = 2$. By definition, $H_2 = Z_2/B_2 = \operatorname{Ker} d_2/\operatorname{Im} d_3$. Since $\operatorname{Im} d_3 = 0$, we have $H_2 \simeq \operatorname{Ker} d_2 = Z_2$. Let

$$x = n_0P_0P_1P_2 + n_1P_0P_1P_3 + n_2P_0P_2P_3 + n_3P_1P_2P_3 \in C_2.$$

Then, by an easy calculation, we get

$$x \in Z_2 \Leftrightarrow d_2(x) = 0 \Leftrightarrow x = n(P_1P_2P_3 - P_0P_2P_3 + P_0P_1P_3 - P_0P_1P_2),$$

where $n \in \mathbb{Z}$. It follows that $Z_2 = \operatorname{Ker} d_2 = (\mathbb{Z}, +)$, and $H_2 = Z_2/B_2 = \mathbb{Z}/\{0\} \simeq (\mathbb{Z}, +)$.

Finally, $H_i(S^2) = \{0\}$, $\forall i \in \mathbb{Z} \setminus \{0, 1, 2\}$.

Exercise 3 Compute the homology with coefficients in $\mathbb{Z}$ of the simplicial sets corresponding to the following triangulated spaces:

a) one point; union of two points; segment;
b) triangle;
c) the 1-dimensional sphere S^1.
d) solid tetrahedron;
e) the plane annular region between two concentric circles;
f) two tangent 1-spheres;
g) two tangent 2-spheres;
h) the cylinder;
g) the Möbius strip;
h) the 2-dimensional torus.
i) the Klein bottle;
j) the real projective plane;
k) a circle touching a 2-sphere at one point;
l) a 2-sphere with an annular ring whose inner circle is a great circle of the 2-sphere;
m) the 2-sphere with two handles;
n) a 2-sphere touching a Klein bottle at one point;
o) the n-dimensional simplex Δ_n;
p) the $(n-1)$-sphere S^{n-1} (i.e. the boundary of Δ_n).

Exercise 4 Let X be a triangulated space. Prove that $H_0(X)$ is isomorphic to the free abelian group generated by the connected components of X.

Exercise 5 Let X and Y be disjoint triangulated spaces. Prove that $H_i(X \cup Y) \simeq H_i(X) \oplus H_i(Y)$.

Example 1.4.5 (singular homology) Let Y be a topological space. By definition, a **singular n-simplex** of Y is a continuous map $\phi : \Delta_n \to Y$. The following data defines a simplicial set.

- Let X_n be the set of all singular n-simplices of Y, for $n \in \mathbb{N}$;
- $X_{(f)}(\phi) = \phi \circ \Delta_f$, where $f : [m] \to [n]$.

Let $C_n(X, A)$ be the group of n-chains of X. Its homology is denoted $H_n^{\text{sing}}(Y, A)$, and it is called the **singular homology** of Y with coefficients in the abelian group A. Similarly, one defines singular cohomology.

Exercise 6 Verify that X is indeed a simplicial set.

1.5 Coefficient systems

We may construct chains and cochains of a simplicial set using as coefficients objects which are more general then the abelian groups. We introduce two types of coefficient systems: homological and cohomological.

Definition 1.5.1 a) A **homological coefficient system** $\mathcal{A}$ on a simplicial set X consists of a family of abelian groups $(\mathcal{A}_x)_{x \in X_n,\, n \in \mathbb{N}}$ and a family of group homomorphisms $(\mathcal{A}_{(f,x)})$, where $x \in X_n$, $n \in \mathbb{N}$, $f : [m] \to [n]$ non-decreasing, and $\mathcal{A}_{(f,x)} : \mathcal{A}_x \to \mathcal{A}_{X(f)x}$, such that the following two conditions are satisfied:

$$\mathcal{A}_{(\text{id},x)} = \text{id}, \qquad \mathcal{A}_{(f \circ g, x)} = \mathcal{A}_{(g, X(f)x)} \circ \mathcal{A}_{(f,x)}.$$

b) A **homological coefficient system** $\mathcal{B}$ on a simplicial set X consists of a family of abelian groups $(\mathcal{B}_x)_{x \in X_n,\, n \in \mathbb{N}}$ and a family of group homomorphisms $(\mathcal{B}_{(f,x)})$, where $x \in X_n$, $n \in \mathbb{N}$, $f : [m] \to [n]$ non-decreasing, and $\mathcal{B}_{(f,x)}\ \mathcal{B}_{X(f)x} \to \mathcal{B}_x$, such that the following two conditions are satisfied:

$$\mathcal{B}_{(\text{id},x)} = \text{id}, \qquad \mathcal{B}_{(f \circ g, x)} = \mathcal{B}_{(f,x)} \circ \mathcal{B}_{(g, X(f)x)}.$$

Definition 1.5.2 a) Let $\mathcal{A}$ be a homological coefficient system on a simplicial set X. An n-dimensional chain of X with coefficients in $\mathcal{A}$ is a formal linear combination $c = \sum_{x \in X_n} a(x)x$, where $a_x \in \mathcal{A}_x$. These chains form an abelian group (under addition) which is denoted by $C_n(X, \mathcal{A})$. The boundary map is

$$d_n : C_n(X, \mathcal{A}) \to C_{n-1}(X, \mathcal{A}), \quad d_n c = \sum_{x \in X_n} \sum_{i=0}^{n} (-1)^i \mathcal{A}_{(\partial_n^i, x)}(a_x) X(\partial_n^i)(x).$$

Exercise 7 Verify that $C_\bullet(X, \mathcal{A})$ is a chain complex, i.e. $d_{n-1} \circ d_n = 0$.

Homology groups of the complex $C_\bullet(X, \mathcal{A})$ are called the homology groups of the simplicial set X with coefficients in $\mathcal{A}$; they are denoted by $H_n(X, \mathcal{A})$.

b) Let $\mathcal{B}$ be a cohomology coefficient system on X. Let $C^n(X, \mathcal{B}) = \{f \mid f : X_n \to \bigcup_{x \in X_n} \mathcal{B}_x \text{ with } f(x) \in \mathcal{B}_x\}$, which is an abelian group under addition. The coboundary operator is

$$d^n : C^n(X, \mathcal{B}) \to C^{n+1}(X, \mathcal{B}), \quad (d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i \mathcal{B}_{(\partial^i_{n+1}, x)}(f(X(\partial^i_{n+1}))x), \quad x \in X_n.$$

Exercise 8 Verify that $C^\bullet(X, \mathcal{B})$ is a chain complex, i.e. $d^{n+1} \circ d^n = 0$.

Cohomology groups of the complex $C^\bullet(X, \mathcal{B})$ are called the cohomology groups of the simplicial set X with coefficients in $\mathcal{B}$; they are denoted by $H^n(X, \mathcal{B})$.

Example 1.5.3 Let A be an abelian group, $\mathcal{A}_x := A$ for all x, and $\mathcal{A}_{(f,x)} = \mathrm{id}$ for all (f, x). This is the **constant coefficient system**, and it is both homological and cohomological. Moreover, we have $H_n(X, \mathcal{A}) = H_n(X, A)$ and $H^n(X, \mathcal{A}) = H^n(X, A)$.

Example 1.5.4 (coverings, Čech cohomology) Let Y be a topological space, $U = (U_\alpha)_{\alpha \in A}$ be a covering of Y (either by open or by closed sets). We define a simplicial set called the **nerve of the covering**.

- Let $X_n = \{(\alpha_0, \dots, \alpha_n) \mid U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \emptyset\}$;
- For $f : [m] \to [n]$, let $X(f)(\alpha_0, \dots, \alpha_n) = (\alpha_{f(0)}, \dots, \alpha_{f(m)})$.

The following data form a cohomological coefficient system:

- $\mathcal{F}_{\alpha_0, \dots, \alpha_n}$ is the group of continuous function (under addition) $U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \to \mathbb{R}$,
- $\mathcal{F}_{(f,(\alpha_0, \dots, \alpha_n))}$ maps a function $\phi : U_{\alpha_{f(0)}} \cap \dots \cap U_{\alpha_{f(m)}} \to \mathbb{R}$ into its restriction to $U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$.

Exercise 9 Verify that:

a) X is a simplicial set;

b) $\mathcal{F}$ is a cohomological coefficient system.

The cohomology groups $H^n(X, \mathcal{F})$ are called the **Čech cohomology groups** of the sheaf of continuous on Y w.r.t the covering U.

The simplicial set X reflects the combinatorial structure of a covering. One can show that if the covering U is locally finite and all nonempty finite intersections $U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$ are contractible, then the geometric realization $|X|$ of X is homotopically equivalent to Y, so that the topology can be efficiently encoded into combinatorical data.

Note that to verify the axioms we need only trivial properties of the restriction of a function to a subset. So, instead of all functions we can take a subset stable under addition and restriction, e.g., smooth functions for a differentiable manifold, analytic functions for a complex manifold, etc. We can also take the group of invertible functions under multiplication.

Example 1.5.5 ((co)homology of groups) Let G be a finite group, and let $(BG)_n = G^n = G \times \dots \times G$ (n times), and for $f : [m] \to [n]$ let $(BG)(f)(g_1, \dots, g_n) = (h_1, \dots, h_m)$, where

$$h_i = \begin{cases} \prod_{j=f(i-1)+1}^{f(i)} g_j, & \text{if } f(i-1) < f(i) \\ e, & \text{if } f(i-1) = f(i). \end{cases}$$

Then BG is a simplicial set, and its geometric realizaton $|BG|$ is called the **classifying space** of G.

In order to define coefficient systems on BG, let A be a $\mathbb{Z}G$-module.

- Let $\mathcal{B}_x = A$ for all $x \in (BG)_n$; next, if $f : [m] \to [n]$, $x = (g_1, \dots, g_n) \in (BG)_n$ and $a \in A$, define $\mathcal{B}_{(f,x)}(a) = ha$, where $h := \prod_{j=1}^{f(0)} g_j$.
- Let $\mathcal{A}_x = A$ for all $x \in (BG)_n$; next, if $f : [m] \to [n]$, $x = (g_1, \dots, g_n) \in (BG)_n$ and $a \in A$, let $\mathcal{A}_{(f,x)}(a) = h^{-1}a$, where $h := \prod_{j=1}^{f(0)} g_j$.

Exercise 10 Verify that:

a) BG is a simplicial set;

b) $\mathcal{A}$ (respectively $\mathcal{B}$) is a (co)homological coefficient system.

The groups $H_n(BG, \mathcal{A})$ (respectively $H^n(BG, \mathcal{B})$) are called the (co)homology groups of G with coefficients in A.

Chapter 2

The long exact sequence

We regard $H_n(X, \mathcal{A})$ and $H^n(X, \mathcal{B})$ as functions of two variables. In some cases these groups can be computed directly. In general, the main techniques to compute these groups apply only in the study of their behavior under the change of X or the change of $\mathcal{A}$ and $\mathcal{B}$.

2.1 Exact Sequences

Definition 2.1.1 a) An **exact sequence** (also called acyclic complex) of abelian groups is a complex

$$C^\bullet = (\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \dots)$$

with $\operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$ $\forall n \in \mathbb{Z}$, or equivalently, $H^n(C^\bullet) = 0$ $\forall n \in \mathbb{Z}$.

b) A **short exact sequence** is an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

Note that this means:

- i is injective ($A \simeq \operatorname{Im} i$)
- p is surjective ($\operatorname{Im} p = C$)
- $\operatorname{Im} i = \operatorname{Ker} p$ (that is, by the 1st Isomorphism Theorem, p and i induce the isomorphism $C \simeq B/A$).

Remark 2.1.2 1) The sequence $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$ is exact, where

- A and C with are abelian groups;
- $A \oplus C = A \times C = \{(a, c) : a \in A, c \in C\}$;
- $i(a) = (a, c)$ (hence $A \simeq A \times \{0\}$);
- $p(a, c) = c$ (hence $C \simeq \{0\} \times C$).

Clearly, $\operatorname{Im} i = A \times \{0\} = \operatorname{Ker} p$.

2) A short exact sequence is also called an **extension** of A by C (or of C by A). The previous remark shows that there is an extension of A by C, but there can be several non-isomorphic extensions. For instance, there are two non-isomorphic extensions of $(\mathbb{Z}_2, +)$ by $(\mathbb{Z}_2, +)$.

(1) $\quad 0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$, where $i : \bar{a} \to (\bar{a}, \bar{a})$, $\quad p : (\bar{a}, \bar{c}) \to \bar{c}$;

(2) $\quad 0 \to \mathbb{Z}_2 \xrightarrow{i} \mathbb{Z}_4 \xrightarrow{p} \mathbb{Z}_2 \to 0$, where $i(\bar{a}) = 2\hat{a}$, $\quad p(\hat{b}) = \bar{b}$.

Clearly $\mathbb{Z}_2 \times \mathbb{Z}_2 \neq \mathbb{Z}_4$.

3) Let $f : A \to B$ be a homomorphism of abelian groups. Then we have the exact sequence

$$0 \to \operatorname{Ker} f \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{p} B/\operatorname{Im} f \to 0,$$

where i is the inclusion and p is the canonical projection.

Exercise 11 Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence. Then the following statements are equivalent:

(i) there is a homomorphism $s : C \to B$ such that $p \circ s = 1_C$;

(ii) there is a homomorphism $r : B \to A$ such that $r \circ i = 1_A$;

(iii) there is an isomorphism $B \simeq A \oplus C$ induced by i and p.

In this case the above sequence is called a **split sequence**.

Theorem 2.1.3 *Let X be a simplicial set and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Then we have the following exact sequences (called* long exact sequences*):*

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C) \to H^1(X, A) \to \cdots \to H^n(X, A) \to H^n(X, B) \to H^n(X, C) \to H^{n+1}(X, A) \to \ldots$$

$$\cdots \to H_n(X, A) \to H_n(X, B) \to H_n(X, C) \to H_{n-1}(X, A) \to \cdots \to H_1(X, C) \to H_0(X, A) \to H_0(X, B) \to H_0(X, C) \to 0$$

The proof of the theorem consists of two main steps:

(1) The construction of all homomorphisms;

(2) The proof of the exactness.

It is convenient to perform both steps in a more general setting. For this, we introduce new concepts.

2.2 Morphisms of Complexes

Let $B^\bullet, C^\bullet$ be complexes.

Definition 2.2.1 A morphism $f^\bullet : B^\bullet \to C^\bullet$ of complexes is a family $f^\bullet = (f^n)_{n\in\mathbb{Z}}$

$$\begin{tikzcd}
\cdots \arrow[r] & B^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & B^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & B^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "d^{n-1}"'] & C^n \arrow[r, "d^n"'] & C^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that for all $n \in \mathbb{Z}$

$$f^{n+1} \circ d^n = d^{n-1} \circ f^n.$$

Remark 2.2.2 A morphism $f^\bullet : B^\bullet \to C^\bullet$ of complexes induces a family $H^n(f^\bullet) : H^n(B) \to H^n(C)$ of homomorphism of abelian groups, where for each $[x] = x + B_n(C_\bullet) \in H_n(C_\bullet)$, with $x \in \operatorname{Ker} d_n$,

$$H_n(f_\bullet)([x]) = [f_n(x)].$$

Indeed, let $[b] \in H^n(B^\bullet)$, that is $b \in B^n$, $d^n(b) = 0$, $[b] = b + B^n(B^\bullet)$. We have $f^n(b) \in C^n$, $d^n(f^n(b)) = f^{n+1}(d^n(b)) = 0$. It follows that $f^n(b) \in Z^n(C^\bullet)$ is a cocycle.

By definition, let $H^n(f)([b]) = [f^n(b)] = f^n(b) + B^n(C^\bullet)$. This is a good definition, that is it does not depend on the choice of representatives. Indeed, let $b' \in Z^n(B^\bullet)$ s.t.

$$[b] = [b'] \Leftrightarrow b - b' \in B^n(B^\bullet) = \operatorname{Im}(d_B^{n-1}) \Leftrightarrow \exists b'' \in B^{n-1} : b - b' = d^{n-1}(b'')$$

We have to see that $[f^n(b)] = [f^n(b')] \Leftrightarrow f^n(b) - f^n(b') \in B^n(C^\bullet)$. Indeed

$$f^n(b) - f^n(b') = f^n(b - b') = f^n\left(d_B^{n-1}(b'')\right) = d_C^{n-1}\left(f^{n-1}(b'')\right) \in \operatorname{Im} d^{n-1}.$$

Finally $H^n(f^\bullet)$ is a homomorphism of abelian groups:

$$\begin{aligned} H^n(f^\bullet)([b] + [b']) &= H^n(f^\bullet)([b + b']) = [f^n(b + b')] = [f^n(b) + f^n(b')] \\ &= [f^n(b)] + [f^n(b')] = H^n(f^\bullet)([b]) + H^n(f^\bullet)([b']). \end{aligned}$$

Definition 2.2.3 a) If $f^\bullet : B^\bullet \to C^\bullet$ is a morphism of complexes, then $\operatorname{Ker} f^\bullet := (\operatorname{Ker} f^n)_{n\in\mathbb{Z}}$, and $\operatorname{Coker} f^\bullet := (\operatorname{Coker} f^n)_{n\in\mathbb{Z}}$.

b) We say that

$$0 \to A^\bullet \xrightarrow{i^\bullet} B^\bullet \xrightarrow{p^\bullet} C^\bullet \to 0$$

is a **short exact sequence of complexes**, if for all $n \in \mathbb{Z}$, the sequence

$$0 \to A^n \xrightarrow{i^n} B^n \xrightarrow{p^n} C^n \to 0$$

of abelian group is exact.

Example 2.2.4 (The mapping cone) 1) Let $f_\bullet : B_\bullet \to C_\bullet$ be a morphism of chain complexes. We define the complex $\mathrm{Cone}(f)_\bullet = (\mathrm{Cone}(f)_n)_{n\in\mathbb{Z}}$, where $\mathrm{Cone}(f^n) = B_{n-1} \oplus C_n$, and

$$\cdots \to \mathrm{Cone}(f)_n \xrightarrow{d_n} \mathrm{Cone}(f)_{n-1} \to \ldots, \quad d_n(b, c) = (-d(b), d(c) - f(b)),$$

i.e.

$$d_n = \begin{bmatrix} -d^B & 0 \\ -f & d^C \end{bmatrix}.$$

Dually, if $f^\bullet : B^\bullet \to C^\bullet$ is a morphism of cochain complexes, we define the cochain complex $\mathrm{Cone}\,(f)^\bullet = (\mathrm{Cone}(f)^n)_{n\in\mathbb{Z}}$ where $\mathrm{Cone}\,(f^n) = B_{n+1} \oplus C_n$

2) **(The mapping cylinder)** Let $f_\bullet : B_\bullet \to C_\bullet$ be a morphism of chain complexes. We define the complex $\mathrm{Cyl}\,(f)_\bullet = (\mathrm{Cyl}(f)_n)_{n\in\mathbb{Z}}$ where $\mathrm{Cyl}\,(f^n) = B_n \oplus B_{n-1} \oplus C_n$, and

$$d_n : \mathrm{Cyl}(f)_n \to \mathrm{Cyl}(f)_{n-1}, \quad d_n(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b')),$$

i.e.

$$d = \begin{bmatrix} d_B & 1_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix}.$$

Exercise 12 (The 3×3 lemma) Consider the commutative diagram with exact colums:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

a) If the first two rows are exact, then the third row is also exact.

b) If the last two rows are exact, then the first row is also exact.

c) If the first and the third rows are exact and the composed map $A \to C$ is zero, then the second row is also exact.

2.3 Homotopic mappings of complexes

In addition to the long exact sequence, there exists another important tool to compute homology: change of complexes that preserve homology.

Let $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ be morphisms of chain complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'"] & C'_n \arrow[r, "d'"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.3.1 a) We say that $f_\bullet$ and $g_\bullet$ are **homotopic** ($f_\bullet \sim g_\bullet$) if there is $k_\bullet = (k_n : C_n \to C'_{n+1})_{n\in\mathbb{Z}}$ such that $f_n - g_n = k_{n-1} \circ d_n + d'_{n+1} \circ k_n$ for all $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "k_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "k_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'"'] & C'_n \arrow[r, "d'"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

b) Similarly, if $f^\bullet, g^\bullet : C^\bullet \to C'^\bullet$ are morphisms of cochain complexes, then we say that $f_\bullet$ and $g_\bullet$ are homotopic if $\exists\, k = (k^n : C^n \to C'^{n-1})_{n\in\mathbb{Z}}$ such that $f^n - g^n = d \circ k + k \circ d$.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "d"] & C^n \arrow[r, "d"] \arrow[dl, "k^n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C^{n+1} \arrow[r] \arrow[dl, "k^{n+1}"] & \cdots \\
\cdots \arrow[r] & C'^{n-1} \arrow[r, "d'"'] & C'^n \arrow[r, "d'"'] & C'^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 13 Prove that homotopy is an equivalence relation.

Lemma 2.3.2 *If* $f_\bullet$ *and* $g_\bullet$ *are homotopic, then* $H_n(f_\bullet) = H_n(g_\bullet), \forall n \in \mathbb{Z}$.

Proof. Let $x \in Z_n(C_\bullet)$, so $d_n x = 0$. Since $f^\bullet \sim g^\bullet$, we have

$$f_n(x) - g_n(x) = (d'_{n+1} \circ k_n)(x) + (k_{n-1} \circ d_n)(x) = d'_{n+1} k_n x \in B_n(C^\bullet).$$

It follows that

$$\begin{aligned}(H_n(f_\bullet) - H_n(g_\bullet))[x] &= H_n(f_\bullet)[x] - H_n(g_\bullet)[x] = [f_n(x)] - [g_n(x)] \\ &= [f_n(x) - g_n(x)] = [(f_n - g_n)(x)] = [d'_{n+1}(k_n(x))] = [0]\end{aligned}$$

Consequently, $H_n(f_\bullet) = H_n(g_\bullet)$. ∎

Definition 2.3.3 Let $C_\bullet$ and $C'_\bullet$ be chain complexes. We say that $C_\bullet$ and $C'_\bullet$ are **homotopic** if there exist a morphism of complexes $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C_\bullet \to C'_\bullet$ such that

$$g_\bullet \circ f_\bullet \sim 1_{C_\bullet}, \quad f_\bullet \circ g_\bullet \sim 1_{C'_\bullet}$$

Exercise 14 Prove that homotopy is an equivalence relation between complexes.

Corollary 2.3.4 *If* $C_\bullet$ *and* $C'_\bullet$ *are homotopic, then* $H_n(C_\bullet) \simeq H_n(C'_\bullet), \forall n \in \mathbb{Z}$.

Proof. We know that $g_\bullet \circ f_\bullet \sim 1_{C_\bullet}$ and $f_\bullet \circ g_\bullet \sim 1_{C'_\bullet}$. We take homology on both sides and so we have the two relations

$$H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet) \quad \text{and} \quad H_n(1_{C_\bullet}) = 1_{H_n(C_\bullet)}.$$

Hence, $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$ is an isomorphism of abelian groups, with $H_n(g_\bullet), \forall n \in \mathbb{Z}$. ∎

Remark 2.3.5 a) There is a concept of homotopy in topology. Let φ and $\psi : X \to Y$ be two continuous maps of topological spaces, and denote $I := [0, 1]$. Then φ and psi are said to be **homotopic** ($\varphi \sim \psi$) if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = \psi(x)$ for all $x \in X$.

Assume that X, I are triangulated spaces and let φ and ψ be continuous maps. Then φ and ψ induce morphism of complexes $\varphi_\bullet, \psi_\bullet : C_\bullet(X) \to C_\bullet(Y)$. If $\varphi \sim \psi$, then $\varphi_\bullet \sim \psi_\bullet$.

b) The topological spaces X and Y are called **homotopic** ($X \sim Y$), if $\exists\, \varphi : X \to Y$ and $\psi : Y \to X$ continuous maps such that $\psi \circ \varphi \sim \mathrm{id}_X$ and $\varphi \circ \psi \sim \mathrm{id}_Y$. Note that if $X \simeq Y$ (i.e. X and Y are homeomorphic), then $X \sim Y$.

If $X \sim Y$, then $H_n(X) \simeq H_n(Y)$ $\forall n \in \mathbb{N}^*$. Equivalently, if $H_\bullet(X) \not\simeq H_\bullet(Y)$, then $X \not\sim Y$.

Chapter 3

Constructions with modules and algebras

We want to construct more examples of complexes, and we shall consider homomorphisms and tensor products in connection with exact sequences and complexes.

3.1 Review of modules

Throughout R will be a ring with 1. Recall that a **left** (respectively **right**) R-**module** is an abelian group $(M,+)$ together with a multiplication with scalars $R\times M\to M$ (respectively $M\times R\to M$) such that the following axioms hold:

$$\begin{array}{c} \alpha(x+y)=\alpha x+\beta y \\ (\alpha+\beta)x=\alpha x+\beta y \\ (\alpha\beta)x=\alpha(\beta x) \\ 1\cdot x=x, \end{array} \qquad \text{respectively} \qquad \begin{array}{c} (x+y)\alpha=x\alpha+y\beta \\ x(\alpha+\beta)=x\alpha+y\beta \\ x(\alpha\beta)=(x\alpha)\beta \\ x\cdot 1=x, \end{array}$$

for all $\alpha,\beta\in R$ and $x,y\in M$. By an R-module we shall mean a left R-module.

Let R and S be two rings. An (R,S)-**bimodule** is an abelian group $(M,+)$ such that M is a left R-module, M is a right S-module, and

- for all $r\in R$, $s\in S$, $x\in M$ we have $(rx)s=r(xs)$.

We denote by ${}_RM$ respectively M_R to emphasize that M is a left respectively right R-module, and ee denote an (R,S)-bimodule by ${}_RM_S$.

A map $g:M\to M'$ is called R-**linear** or a **homomorphism** of R-modules if

$$g(\alpha x+\beta y)=\alpha g(x)+\beta g(y)$$

for all $\alpha,\beta\in R$ and $x,y\in M$.

A nonempty subset N of M is a R-**submodule**, if it is closed under both operations. It is easy to see that if $g:M\to M'$ is an R-homomorphism, then the **kernel**

$$\operatorname{Ker} f:=\{x\in M\mid f(x)=0\}$$

of f is an R-submodule of M, and the **image**

$$\operatorname{Im} f:=\{f(x)\mid x\in M\}$$

is an R-submodule of M'.

If R is a field, then an R-module is a called an R-vector space. Note also that the concepts of abelian group $\mathbb{Z}$-module coincide.

A subset X of a (left) R-module M is called a **basis** of M is every map $f:X\to N$, where N is another R-module, can be uniquely extended to a R-linear map $\overline{f}:X\to N$. An R-module which has a basis is called **free**. Any vector space is a free module, but in general, not any R-module is free.

More intuitively, R is a free R-module with the basis X if and only if any element of M can be expressed in a unique way as a linear combination of finitely many elements of X. In this case,

$$M=\{\sum_{x\in X}\alpha_x x\mid \alpha_x\in R,\ \alpha_x=0 \text{ for all but finitely many } x\in X\}.$$

Exercise 15 Let ${}_RM$, ${}_RM'$ and ${}_RF$ be R-modules. Assume that $f : M \to M'$ and $p : M \to F$ are R-homomorphisms, and that p is surjective. Consider the following diagram.

$$\begin{tikzcd} & M' . \\ F \arrow[ur, dotted, "\overline{f}"] & M \arrow[l, "p"'] \arrow[u, "f"'] \end{tikzcd}$$

a) There is an R-homomorphism $\overline{f} : F \to M'$ such that $\overline{f} \circ p = f$ if and only if $\operatorname{Ker} p \subseteq \operatorname{Ker} f$. Moreover, in this case, $\overline{f}$ is unique;
b) $\overline{f}$ is injective if and only if $\operatorname{Ker} p = \operatorname{Ker} f$;
c) $\overline{f}$ is surjective if and only if f is surjective.

3.2 The group of R-homomorphisms

Let R be a ring (associative, with unit). If M, and N are (left) R-modules, then

$$\mathrm{Hom}_R(M, N) := \{f : M \to N \mid f(x + x') = f(x) + f(x'),\ f(rx) = rf(x),\ \forall r \in R,\ x, x' \in M\},$$

is the set of R-linear maps. In fact $(\mathrm{Hom}_R(M, N), +)$ is an abelian group, where

$$f + g : M \to N, \quad (f + g)(x) = f(x) + g(x).$$

Proposition 3.2.1 *Let* R, S, T *be rings.*
a) *If* ${}_RM_S$, ${}_RN_T$ *are bimodules, then* $\mathrm{Hom}_R(M, N)$ *is an* (S, T)*-bimodule;*
b) *If* ${}_SM_R$, ${}_TN_R$ *are bimodules, then* $\mathrm{Hom}_R(M, N)$ *is a* (T, S)*-bimodule.*

Proof. a) For $s \in S, t \in T$, and $f : M \to N$ left R-linear, define

$$sf : M \to N, \quad (sf)(x) = f(xs),$$
$$ft : M \to N, \quad (ft)(x) = f(x)t.$$

For $x, y \in M$, $(sf)(x + y) = f((x + y)s) = f(xs + ys) = f(xs) + f(ys) = (sf)(x) + (sf)(y)$, from which it follows that sf is a homomorphism. Furthermore, for $r \in R$ $(sf)(rx) = f((rx)s) = f(r(xs)) = rf(xs) = r(sf)(x)$, thus $sf \in \mathrm{Hom}_R(M, N)$.

$(ft)(x + y) = f(x + y)t = (f(x) + f(y))t = f(x)t + f(y)t = (ft)(x) + (ft)(y)$, hence ft is a homomorphism. Furthermore, $(ft)(rx) = f(rx)t = (rf(x))t = r(f(x)t) = r(ft)(x)$, thus $tf \in \mathrm{Hom}_R(M, N)$.

Let $\cdot : S \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N)$ given by $(s, f) \mapsto sf$ and "$\cdot$" $: \mathrm{Hom}_R(M, N) \times T \to \mathrm{Hom}_R(M, N)$ given by $(f, t) \mapsto ft$. We can easily check that $\mathrm{Hom}_R(M, N)$ is a left S-module and a right T-module. We have the maps $(sf)t$, $s(ft) : M \to N$, where

$$[(sf)t](x) = (sf)(x) \cdot t = f(sx) \cdot t, \quad [s(ft)](x) = (ft)(sx) = f(sx) \cdot t.$$

Hence $(sf)t = s(ft)$.

The proof of b) is similar. ■

Remark 3.2.2 a) Fix ${}_RM$ and regard $\mathrm{Hom}_R(M, -)$ as a "function" of one "variable". The "variable" can be another module as an R-linear map.

If N is an *R*-module, then $(\mathrm{Hom}_R(M, N), +)$ was define above. Let $f : M \to N'$ be an R-linear map. Then by definition

$$\mathrm{Hom}_R(M, f) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'), \quad \mathrm{Hom}_R(M, f)(\alpha) = f \circ \alpha,$$

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr, "f\circ\alpha"'] & N \arrow[d, "f"] \\ & N' \end{tikzcd}$$

We get a group homomorphism, since

$$\mathrm{Hom}_R(M, f)(\alpha + \beta) = f \circ (\alpha + \beta) = f \circ \alpha + f \circ \beta = \mathrm{Hom}_R(M, f)(\alpha) + \mathrm{Hom}_R(M, f)(\beta).$$

b) We can similarly regard $\mathrm{Hom}_R(-, M)$ as a "function". If N is an R-module, then $\mathrm{Hom}_R(N, M), +)$ is defined as above. Let $f : N \to N'$ be an R-linear map. Then

$$\mathrm{Hom}_R(f, M) : \mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N, M), \quad \mathrm{Hom}_R(f, M)(\alpha) = \alpha \circ f,$$

$$\begin{tikzcd} N' \arrow[r, "\alpha"] & M \\ N \arrow[u, "f"] \arrow[ur, "\alpha\circ f"'] & \end{tikzcd}$$

This also gives a group homomorphisms.

3.3 The tensor product

Definition 3.3.1 Let ${}_RN$ and M_R be R-modules, and G an abelian group. Then a map $\varphi : M \times N \to G$ is called **R-balanced** if

$$\begin{aligned}
&\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)\\
&\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)\\
&\varphi(\alpha x, y) = \varphi(x, \alpha y)
\end{aligned}$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and all $\alpha \in R$.

Exercise 16 For any modules ${}_RN$ and M_R and any abelian group G, if $\varphi : M \times N \to G$ is R-balanced and $f : G \to X$ is a group homomorphism, then $f \circ \varphi : M \times N \to X$ is R-balanced.

Proposition 3.3.2 *For any two R-modules ${}_RN$ and M_R there is an abelian group G and an R-balanced map $\varphi : M \times N \to G$ satisfying the following universal property.*

$$\begin{tikzcd}
 & G \arrow[dl, dotted, "f"'] \\
X & M \times N \arrow[l, "\psi"] \arrow[u, "\varphi"']
\end{tikzcd}$$

For any abelian group $(X, +)$ and any R-balanced map $\psi : M \times N \to X$ there is a unique group homomorphism $f : G \to X$ such that $\psi = f \circ \varphi$. Moreover this universal property determines G up to an isomorphism.

Proof. Let L be the free abelian group (i.e. free $\mathbb{Z}$-module with basis $M \times N$. This means that

$$L = \left\{ \sum_{(x,y)\in M\times N} a_{(x,y)}(x, y) \mid a_{(x,y)} \in \mathbb{Z},\ a_{(x,y)} = 0 \text{ for almost all } (x, y) \in M \times N \right\}.$$

Let H be the subgroup of L generated by the elements

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y),\ (x, y_1 + y_2) - (x, y_1) - (x, y_2),\ (x\alpha, y) - (x, \alpha y),$$

where $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, $\alpha \in R$. We set $G = L/H$ and let $\varphi = p \circ i : M \times N \to G$, where $M \times N \xrightarrow{i} L \xrightarrow{p} L/H = G$ are the inclusion map, respectively the canonical projection.

Let X be an abelian group and $\psi : M \times N \to X$ an R-balanced map. As $M \times N$ is a basis for L, we get a unique group homomorphism $g : L \to G$ such that $g \circ i = \psi$. In fact,

$$g\left(\sum_{(x,y)\in M\times N} a_{(x,y)}(x, y) \right) = \sum_{(x,y)\in M\times N} a_{(x,y)}\psi(x, y).$$

It is easy to show that $\operatorname{Ker} p = H$ and $H \subseteq \operatorname{Ker} g$. So, due to Exercise 15, there is a unique $f : G \to X$ such that $f \circ p = g$. Therefore $f \circ \varphi = f \circ p \circ i = g \circ i = \psi$, and f is unique with this property.

Finally, Let X be an abelian group and ψ a balanced map satisfying the same universal property as G and φ.

$$\begin{tikzcd}
 & G \arrow[dl, dotted, "f"'] \\
X & M \times N \arrow[l, "\psi"] \arrow[u, "\varphi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G \\
X \arrow[ur, dotted, "f'"] & M \times N \arrow[l, "\psi"] \arrow[u, "\varphi"']
\end{tikzcd}$$

Then $f' \circ f \circ \varphi = f' \circ \psi = \varphi$, and $f \circ f' \circ \psi = \varphi$. It follows that $f' \circ f = \mathrm{id}_G$, and $f \circ f' = \mathrm{id}_X$. ∎

Definition 3.3.3 The group constructed above is called the **tensor product** of M and N over R, and is denoted by $M \otimes_R N$. We also denote $x \otimes y := \varphi(x, y)$, where $\varphi : M \times N \to M \otimes_R N$ is the canonical R-balanced map.

Remark 3.3.4 The set $\{x \otimes y \mid x \in M, y \in N\}$ generates $M \otimes_R N$ as an abelian group, that is, for each $z \in M \otimes_R N$ there are $r \geq 1$, $x_1, \ldots, x_r \in M$, $y_1, \ldots, y_r \in N$, such that $z = x_1 \otimes y_1 + \cdots + x_r \otimes y_r$. Note that such an expression for z is not unique.

Let $f : M_R \to M_{R'}$ and $g : {}_RN \to {}_RN'$ be R-linear maps and denote:

$$\begin{aligned}
&M \otimes_R g : M \otimes_R N \to M \otimes_R N', \quad M \otimes_R g = \mathrm{id}_M \otimes g\\
&f \otimes_R N : M \otimes_R N \to M' \otimes_R N, \quad f \otimes_R N = f \otimes \mathrm{id}_N.
\end{aligned}$$

Lemma 3.3.5 a) *If* g *is surjective then* $M \otimes_R g$ *is surjective.*
b) *If* f *is surjective then* $f \otimes_R N$ *is surjective.*

Remark 3.3.6 If g is injective, it does not follow that $M \otimes_R g$ is injective. As an example, let $R = \mathbb{Z}$, $I : \mathbb{Z} \to \mathbb{Q}$ and $M = \mathbb{Z}_2$. We see that $M \otimes \mathbb{Z} \simeq M$ for all abelian groups M, but $\mathbb{Z}_2 \otimes \mathbb{Q} = 0$.

Definition 3.3.7 We say that the right R-module M is **flat** if $M \otimes_R g$ is injective whenever $g : N \to N'$ is an injective linear map (of left R-modules).

Proposition 3.3.8 *Let* R, S, T *be rings and let* ${}_SM_R$, ${}_RN_T$ *be bimodules. Then* $M \otimes_R N$ *is an* (S, T)*-bimodule.*

Proof. We define "$\cdot$" : $S \times (M \otimes_R N) \to M \otimes_R N$ induced by

$$(s, x \otimes y) \mapsto s(x \otimes y) = (sx) \otimes y$$

and "$\cdot$" : $(M \otimes_R N) \times T \to M \otimes_R N$ induced by

$$(x \otimes y, t) \mapsto (x \otimes y)t = x \otimes (yt),$$

where $M \otimes_R N = \{x \otimes y | x \in M, y \in N\}$.

For $s \in S$ let $\psi_s : M \times N \to M \otimes_R N$ given by $\psi_s(x, y) = (sx) \otimes y$. We check that ψ_s is balanced. We have $\psi_s(x_1 + x_2, y) = (s(x_1 + x_2) \otimes y) = (sx_1 + sx_2) \otimes y = sx_1 \otimes y + sx_2 \otimes y = \psi_s(x_1, y) + \psi_s(x_2, y)$. Also, $\psi_s(sx, y_1 + y_2) = (sx) \otimes (y_1 + y_2) = (sx) \otimes y_1 + (sx) \otimes y_2 = \psi_s(x, y_1) + \psi_s(x, y_2)$. Finally $\psi_s(xr, y) = s(xr) \otimes y = (sx)r \otimes y = (sx) \otimes ry = \psi_s(x, ry)$, showing that ψ_s is balanced. It follows that there exists an f_s group isomorphism on $M \otimes_R N$. Then, for $z \in M \otimes_R N$, $sz = f_s(z)$.

It is easy to show that ${}_SM \otimes_R N_T$ is a bimodule. ■

Lemma 3.3.9 *If* N *is a left* R*-module and* G *is an abelian group then*

$$\mathrm{Hom}_{\mathbb{Z}}(N, G) = \{f : N \to G \mid f \text{ is } \mathbb{Z} \text{ linear }\} = \{f : N \to G \mid f \text{ is a group homomorphism}\}$$

has natural structure of a right R*-module.*

Proof. We define a multiplication with scalars $\mathrm{Hom}_{\mathbb{Z}}(N, G) \times R \to \mathrm{Hom}_{\mathbb{Z}}(N, G)$ by $(f, r) \mapsto fr$ where $fr : N \to G$, $fr(x) = f(rx)$. This multiplication is well defined, since $fr \in \mathrm{Hom}_{\mathbb{Z}}(N, G)$. ■

Theorem 3.3.10 *Let consider a right* R*-module* M *and a left* R*-module* N *and an abelian group* G*. Then there is a group isomorphism*

$$\alpha_{M,N,G} : \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, G)) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, G)$$

which is natural in M, N, G.

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