2.2. The homology resp. cohomology exact sequence

The groups $H_n(X,A)$ and $H^n(X,A)$ where X is a simplicial set and A and B are coefficient systems can be computed directly only in some simple cases. The main technique consists in the study of behavior of these groups under the change of X or the change of A. In this section we want to prove the following:

Theorem 2.2.1 Let X be a simplicial set and let

$$0 \to A \xrightarrow{i} B \xrightarrow{f} C \to 0$$

be a short exact sequence of abelian groups. These data determine a cohomology exact sequence:

$$0 \to H^0(X,A) \to H^0(X,B) \to H^0(X,C) \to H^1(X,A) \to \dots$$

$$\to H^n(X,A) \to H^n(X,B) \to H^n(X,C) \to H^{n+1}(X,A) \to \dots$$

and a similar homology exact sequence

$$\dots \to H_n(X,A) \to H_n(X,B) \to H_n(X,C) \to H_{n-1}(X,A) \to \dots$$

$$\to H_1(X,C) \to H_0(X,A) \to H_0(X,B) \to H_0(X,C) \to 0 .$$

Remark 2.2.2 The previous theorem has an obvious special case, namely if the exact sequence of the groups splits, so $B \cong A \oplus C$ (see Prop. 1.3.8). Then the exact sequences from Theorem 2.2.1 can be decomposed into split exact sequences

$$0 \to H^n(X,A) \to H^n(X,B) = H^n(X,A) \oplus H^n(X,C) \to H^n(X,C) \to 0$$

$$0 \to H_n(X,A) \to H_n(X,B) = H_n(X,A) \oplus H_n(X,C) \to H_n(X,C) \to 0$$

Exercise 1: Verify the truth of Remark 2.2.2.

For the proof of the above theorem it is more convenient to work in the more general setting of unbounded complexes of R-modules.

A **complex** of R-modules is a sequence of R-modules and R-linear maps of the form

$$A^\bullet = (\dots \to A^n \xrightarrow{d_A^n} A^{n+1} \to \dots)$$

such that $d_A^{n+1} \cdot d_A^n = 0$ for all $n \in \mathbb{Z}$. Remark that this condition is equivalent to $\operatorname{Im} d_A^n \subseteq \operatorname{Ker} d_A^{n+1}$. The R-linear maps $(d_A^n)_{n\in\mathbb{Z}}$ are called the **differentials** of the complex $A^\bullet$.

A morphism of complexes $f^\cdot : A^\cdot \longrightarrow B^\cdot$ is a family of R-linear maps $f^n : A^n \to B^n$ commuting with differentials, as follows:

$$\begin{tikzcd} A^\cdot = \cdots \arrow[r] \arrow[d, "f^\cdot"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ B^\cdot = \cdots \arrow[r] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r] & \cdots \end{tikzcd} \qquad f^{n+1} \cdot d_A^n = d_B^n \cdot f^n$$

For a morphism of complexes $f^\cdot : A^\cdot \longrightarrow B^\cdot$, let denote

$$\operatorname{Ker} f^\cdot = (\operatorname{Ker} f^n)_{n \in \mathbb{Z}} \qquad (\operatorname{Coker} f^\cdot) = (\operatorname{Coker} f^n)_{n \in \mathbb{Z}}$$

with the differentials induced by differentials in $A^\cdot$ and $B^\cdot$. More precisely, recall that $\operatorname{Coker} f^n = B^n / \operatorname{Im} f^n$ and we have the diagram with exact columns (see Ch. 1, Sect. 1, Ex. 18)

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\ \operatorname{Ker} f^\cdot = (\cdots \arrow[r] \arrow[d] & \operatorname{Ker} f^n \arrow[rr, "d^n_{\operatorname{Ker} f^\cdot}"] \arrow[d] & & \operatorname{Ker} f^{n+1} \arrow[r] \arrow[d] & \cdots \\ A^\cdot = \cdots \arrow[r] \arrow[d, "f^\cdot"'] & A^n \arrow[rr, "d_A^n"] \arrow[d, "f^n"'] & & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ B^\cdot = \cdots \arrow[r] \arrow[d] & B^n \arrow[rr, "d_B^n"] \arrow[d] & & B^{n+1} \arrow[r] \arrow[d] & \cdots \\ \operatorname{Coker} f^\cdot = \cdots \arrow[r] \arrow[d] & \operatorname{Coker} f^n \arrow[rr, "d^n_{\operatorname{Coker} f^\cdot}"'] \arrow[d] & & \operatorname{Coker} f^{n+1} \arrow[r] \arrow[d] & \cdots \\ 0 & 0 & & 0 & \end{tikzcd}$$

Note that the differentials $d^n_{\operatorname{Ker} f^\cdot}$ and $d^n_{\operatorname{Coker} f^\cdot}$ are induced by the universal property of the kernel, resp. cokernel (see Ch. 1., Sect 1. Ex. 18 resp. Ex. 19)

A sequence of complexes and morphisms of complexes $A^\cdot \xrightarrow{f^\cdot} B^\cdot \xrightarrow{g^\cdot} C^\cdot$ is called <u>exact</u> at $B^\cdot$ if all sequences of R-modules $A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n$ $(n \in \mathbb{Z})$ are exact at B. In this sense the ~~first~~ sequence of complexes occurring in the diagram above is exact, i.e. it is exact at each term. As in the case of sequences of R-modules, an exact sequence of complexes of the form $0 \longrightarrow A^\cdot \xrightarrow{f^\cdot} B^\cdot \xrightarrow{g^\cdot} C^\cdot \longrightarrow 0$ is called <u>short exact</u>. The direct sum provides an example of such short exact sequence: $0 \longrightarrow A^\cdot \longrightarrow A^\cdot \oplus C^\cdot \longrightarrow C^\cdot \longrightarrow 0$.

Being given a complex of R-modules

$$A^\bullet = (\cdots A^n \xrightarrow{d_A^n} A^{n+1} \longrightarrow \cdots)$$

we define the following R-modules for all $n \in \mathbb{Z}$:

$$Z^n(A^\bullet) = \operatorname{Ker} d_A^n \leq A^n$$

$$B^n(A^\bullet) = \operatorname{Im} d_A^{n-1} \leq A^n$$

$$H^n(A^\bullet) = Z^n(A^\bullet)/B^n(A^\bullet) = \operatorname{Ker} d_A^n / \operatorname{Im} d_A^{n-1}$$

called the n-th module of cycles, boundaries resp. cohomology. (Recall that the condition $d_A^n \cdot d_A^{n-1} = 0$ is equivalent to $\operatorname{Im} d_A^{n-1} \subseteq \operatorname{Ker} d_A^n$, allowing us to construct $H^n(A^\bullet)$).

For a morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$ of complexes of R-modules let define

$$Z^n(f^\bullet) : Z^n(A^\bullet) \longrightarrow Z^n(B^\bullet), \quad Z^n(f^\bullet)(a) = f^n(a)$$

$$B^n(f^\bullet) : B^n(A^\bullet) \longrightarrow B^n(B^\bullet), \quad B^n(f^\bullet)(a) = f^n(a)$$

$$H^n(f^\bullet) : H^n(A^\bullet) \longrightarrow H^n(B^\bullet), \quad H^n(f^\bullet)(a + \operatorname{Im} d_A^{n-1}) = f^n(a) + \operatorname{Im} d_B^{n-1}$$

<u>Exercice 2</u>: Show that, the formulas above ~~define well~~ induce well defined R-linear maps $Z^n(f^\bullet)$, $B^n(f^\bullet)$, $H^n(f^\bullet)$. Moreover, verify that they are the same maps as those induced by the universal property of $\operatorname{Ker} d_B^n = Z^n(B^\bullet)$, $\operatorname{Im} d_B^{n-1} = B^{n-1}(B^\bullet)$, respectively a suitable factorization through a surjection for $H^n(f^\bullet)$.

Consider now two composable morphisms of complexes of R-modules $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet$. Then it holds

$$Z^n(g \cdot f) = Z^n(g) \cdot Z^n(f)$$

$$B^n(g \cdot f) = B^n(g) \cdot B^n(f)$$

$$H^n(g \cdot f) = H^n(g) \cdot H^n(f)$$

for all $n \in \mathbb{Z}$ (exercice 3)

Lemma 2.2.3 Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups, and let X be a simplicial set. Then the sequence of groups of chains and of cochains

$$0 \to C_\bullet(X,A) \xrightarrow{i_\bullet} C_\bullet(X,B) \xrightarrow{p_\bullet} C_\bullet(X,C) \to 0$$

$$0 \to C^\bullet(X,A) \xrightarrow{i^\bullet} C^\bullet(X,B) \xrightarrow{p^\bullet} C(X,C) \to 0$$

are exact.

Proof An element of $C_n(X,A)$ is a formal linear combination $\sum_{x\in X_n} a(x)x$, with $a(x) \in A$. The image of this element under the map $i_n : C_n(X,A) \to C_n(X,B)$ is $\sum_{x\in X_n} i(a(x))\,x$ and i_n is an injection since i has the same property. Similarly one proves that p_n is a surjection. Further $(p_n \circ i_n)\left(\sum_{x\in X_n} a(x)\,x\right) =$
$= \sum_{x\in X_n} (p\circ i)\,(a(x))\,x = 0$, so $\operatorname{Im} i_n \subseteq \operatorname{Ker} p_n$. Conversely let $\beta = \sum_{x\in X_n} b(x)\,x \in C_n(X,B)$ such that $p_n(\beta) = \sum_{x\in X_n} p(b(x))\,x = 0$.
Then $p(b(x)) = 0$ for all $x\in X_n$ so there are ~~$a(x) \in X_n$~~ $a(x)\in A$ such that ~~$p\,b$~~ $b(x) = i(a(x))$ for all $x \in X_n$. Thus
$\beta = i\left(\sum_{x\in X_n} a(x)\,x\right)$ so $\operatorname{Ker} p_n \subseteq \operatorname{Im} i_n$.
The proof for the sequence of cochains is exercise 4.

Now the proof of our motivating Theorem 2.2.1 will be a consequence of Lemma 2.2.3 and the following:

Theorem 2.2.4. Let $0 \to A^\bullet \xrightarrow{i^\bullet} B^\bullet \xrightarrow{p^\bullet} C^\bullet \to 0$ be an exact sequence of complexes of R-modules. Then for any $n \in \mathbb{Z}$, there is an R-linear map $\delta^n : H^n(C^\bullet) \to H^{n+1}(A^\bullet)$ such that the sequence

$$\cdots \to H^n(A^\bullet) \xrightarrow{H^n(i^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(p^\bullet)} H^n(C^\bullet) \xrightarrow{\delta^n} H^{n+1}(A^\bullet) \to \cdots$$

is exact.

Proof. First we construct $\delta^n = \delta^n(i^\bullet; p^\bullet) : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet)$.

Let $\bar{c} = c + \operatorname{Im} d_C^{n-1} \in H^n(C^\bullet)$, so $c \in C^n$ such that $d_C^n(c) = 0$. Since $p^n : B^n \to C^n$ is surjective, there is $b \in B^n$ s.t. $c = p^n(b)$. It holds:

$$p^{n+1}(d_B^n(b)) = d_C^n(p^n(b)) = d_C^n(c) = 0,$$

so $d_B^n(b) \in \operatorname{Ker} p^{n+1} = \operatorname{Im} i^{n+1}$. Thus $d_B^n(b) = i^{n+1}(a)$ for some (unique) $a \in A^{n+1}$. Note also that

$$i^{n+2}(d_A^{n+1}(a)) = d_B^{n+1}(i^{n+1}(a)) = d_B^{n+1}(d_B^n(b)) = 0,$$

so $d_A^{n+1}(a) = 0$ since i^{n+2} is injective. This shows that $a \in \operatorname{Ker} d_A^{n+1}$, so $\bar{a} = a + \operatorname{Im} d_A^n \in H^{n+1}(A^\bullet)$. We put $\delta^n(\bar{c}) = \bar{a}$. All statements above may be illustrated by the diagram with exact columns and semi-exact rows:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "i^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "i^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "i^{n+1}"'] & A^{n+2} \arrow[d, "i^{n+2}"'] \\
B^{n-1} \arrow[r, "d_B^{n-1}"] \arrow[d, "p^{n-1}"'] & B^n \arrow[r, "d_B^n"] \arrow[d, "p^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"] \arrow[d, "p^{n+1}"'] & B^{n+2} \arrow[d, "p^{n+2}"'] \\
C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d] & C^n \arrow[r, "d_C^n"] \arrow[d] & C^{n+1} \arrow[r, "d_C^{n+1}"] \arrow[d] & C^{n+2} \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

We want to show that the definition of δ^n does not depend on the choice of the representative c of $\bar{c} \in H^n(C^\bullet)$ and of the choice of $b \in B^n$ with the property $p^n(b) = c$. Let $c' \in \operatorname{Ker} d_C^n$ be another representative of $\bar{c}$ and let $b' \in B^n$ with $p^n(b') = c'$. Then $c' - c \in \operatorname{Im} d_C^{n-1}$ so $c' = c + d_C^{n-1}(c_0)$ for some $c_0 \in C^{n-1}$. Since p^{n-1} is surjective, we obtain $b_0 \in B^{n-1}$ s.t. $p^{n-1}(b_0) = c_0$. Thus

$$p^n(b' - b - d_B^{n-1}(b_0)) = p^n(b') - p^n(b) - p^n(d_B^{n-1}(b_0)) = c' - c - d_C^{n-1}(p^{n-1}(b_0)) = c' - c - d_C^{n-1}(c_0) = 0$$

so $b' - b - d_B^{n-1}(b_0) \in \operatorname{Ker} p^n = \operatorname{Im} i^n$, and there is a (unique) $a_0 \in A^n$ s.t. $b' - b - d_B^{n-1}(b_0) = i^n(a_0)$. If $a' \in A^{n+1}$ is the unique element obtained from c' and b' as a from c and b, i.e. $d_B^n(b') = i^{n+1}(a')$, then

$$i^{n+1}(a') = d_B^n(b') = d_B^n\left(b + d_B^{n-1}(b_0) + i^n(a_0)\right) = d_B^n(b) + (d_B^n \circ d_B^{n-1})(b_0) + d_B^n(i^n(a_0))$$

$$\cancel{= i^{n+1}(a) + 0 + i^{n+1} d} = i^{n+1}(a) + 0 + i^{n+1}(d_A^n(a_0)) = i^{n+1}(a + d_A^n(a_0))$$

so $a' = a + d_A^n(a_0)$ since i^{n+1} is injective. Thus $\overline{a'} = \bar{a}$.

The verification that δ^n is an R-linear map is <u>exercise 5</u>.

We have now to show the exactness of the sequence

$$\cdots \to H^n(A^\bullet) \xrightarrow{H^n(i^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(p^\bullet)} H^n(C^\bullet) \xrightarrow{\delta^n} H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(i^\bullet)} H^{n+1}(B^\bullet) \to \cdots$$

We do this in three steps:

a) Exactness at $H^n(B^\bullet)$. Since $H^n(p^\bullet)\cdot H^n(i^\bullet) = H^n(p^\bullet \cdot i^\bullet) = H^n(0) = 0$ we deduce $\operatorname{Im} H^n(i^\bullet) \subseteq \operatorname{Ker} H^n(p^\bullet)$. Conversely, if $\bar b \in \operatorname{Ker} H^n(p^\bullet)$ i.e. $\bar b \in H^n(B^\bullet)$ s.t. $H^n(p^\bullet)(\bar b) = 0$, where $\bar b = b + \operatorname{Im} d_B^{n-1}$ for some $b \in B^n$ with $d_B^n(b) = 0$. From $H^n(p^\bullet)(\bar b) = 0$ follows $p^n(b) \in \operatorname{Im} d_C^{n-1}$ i.e. there is $c \in C^{n-1}$ s.t. $p^n(b) = d_C^{n-1}(c)$. By surjectivity of p^{n-1} we get $b_1 \in B^{n-1}$ s.t. $c = p^{n-1}(b_1)$. Thus $p^n(b - d_B^{n-1}(b_1)) = p^n(b) - (p_n \cdot d_B^{n-1})(b_1) = p^n(b) - (d_C^{n-1} \cdot p^{n-1})(b_1)$
$= d_C^{n-1}(c) - d_C^{n-1}(p^{n-1}(b_1)) = d_C^{n-1}(c) - d_C^{n-1}(c) = 0$, so $b - d_B^{n-1}(b_1) \in \operatorname{Ker} p^n = \operatorname{Im} i^n$.
We obtain an element $a \in A^n$ s.t. $b - d_B^{n-1}(b_1) = i^n(a)$. Moreover $(i^{n+1} \cdot d_A^n)(a) =$
$= (d_B^n \cdot i^n)(a) = d_B^n(b - d_B^{n-1}(b_1)) = \underbrace{d_B^n(b)}_{0} - \underbrace{(d_B^n \cdot d_B^{n-1})}_{0}(b_1) = 0$, so $d_A^n(a) = 0$ since i^{n+1} is injective. This shows that $a \in \operatorname{Ker} d_A^n$, so $\bar a = a + \operatorname{Im} d_A^{n-1} \in H^n(A^\bullet)$.
Finally $H^n(i^\bullet)(\bar a) = \overline{i^n(a)} = i^n(a) + \operatorname{Im} d_B^{n-1} = b - d_B^{n-1}(b_1) + \operatorname{Im} d_B^{n-1} = b + \operatorname{Im} d_B^{n-1} = \bar b$, therefore $\bar b \in \operatorname{Im} H^n(i^\bullet)$, showing that $\operatorname{Ker} H^n(p^\bullet) \subseteq \operatorname{Im} H^n(i^\bullet)$.

b). Exactness at $H^n(C^\bullet)$. First we shall show that $\operatorname{Im} H^n(p^\bullet) \subseteq \operatorname{Ker} \delta^n$, or equivalently $\delta^n \cdot H^n(p^\bullet) = 0$. Let $\bar b = b + \operatorname{Im} d_B^{n-1} \in H^n(B^\bullet)$. By definition of δ^n we have $(\delta^n \cdot H^n(p^\bullet))(\bar b) = \delta^n(H^n(p^\bullet)(\bar b)) = \delta^n(p^n(b) + \operatorname{Im} d_C^{n-1}) = \bar a$
$= a + \operatorname{Im} d_A^n$, where $a \in A^{n+1}$ is the unique element for which $i^{n+1}(a) = d_B^n(b)$.
But $b \in \operatorname{Ker} p^n$ since $\bar b = b + \operatorname{Im} d_B^{n-1} \in H^n(B^\bullet)$, so there is $a_1 \in A^n$ s.t. $b = i^n(a_1)$. It follows $i^{n+1}(d_A^n(a_1)) = (d_B^n \cdot i^n)(a_1) = d_B^n(b) = i^{n+1}(a)$, so $a = d_A^n(a_1)$ since i^{n+1} is injective. Therefore $a \in \operatorname{Im} d_A^n$ and $\bar a = 0$ in $H^{n+1}(A^\bullet)$.
Conversely, let $\bar c \in \operatorname{Ker} \delta^n$ i.e. $\bar c = c + \operatorname{Im} d_C^{n-1}$ with $d_C^n(c) = 0$ such that $\delta^n(\bar c) = 0$. By the construction of δ^n, we have $c = p^n(b)$ and $d_B^n(b) = i^{n+1}(a)$ for some $b \in B^n$ and $a \in A^{n+1}$, and $\delta^n(\bar c) = \bar a$. Since $\delta^n(\bar c) = 0$ in $H^{n+1}(A^\bullet)$ we deduce $a \in \operatorname{Im} d_A^n$ i.e. there is $a_1 \in A^n$ s.t. $d_A^n(a_1) = a$. For ~~$b_1 = b - i^n(a_1) \in B^n$~~
we have $d_B^n(b_1) = d_B^n(b) - (d_B^n \cdot i^n)(a_1) = d_B^n(b) - (i^{n+1} \cdot d_A^n)(a_1) = d_B^n(b) - i^{n+1}(a) = 0$ $b_1 = b - i^n(a_1) \in B^n$

so $b_1 \in \operatorname{Ker} d_B^n$ and $\bar{b}_1 = b_1 + \operatorname{Im} d_B^{n-1} \in H^n(B^\cdot)$. Moreover

$$H^n(p^\cdot)(\bar{b}_1) = \overline{p^n(b_1)} = p^n(b_1) + \operatorname{Im} d_C^{n-1} = p^n\big(b - i^n(a_1)\big) + \operatorname{Im} d_C^{n-1} = p^n(b) - \underbrace{(p^n \cdot i^n)}_{0}(a_1) + \operatorname{Im} d_C^{n-1}$$

$$= p^n(b) + \operatorname{Im} d_C^{n-1} = c + \operatorname{Im} d_C^{n-1} = \bar{c}, \text{ so } \operatorname{Ker}\delta^n \subseteq \operatorname{Im} H^n(p^\cdot).$$

d) Exactness at $H^{n+1}(A)$. As before, first we show that $\operatorname{Im}\delta^n \subseteq \operatorname{Ker} H^{n+1}(i^\cdot)$ by verifying that $H^{n+1}(i^\cdot)\cdot \delta^n = 0$. Let $\bar{c} = c + \operatorname{Im} d_C^{n-1} \in H^n(C^\cdot)$ i.e. $d_C^n(c) = 0$. By construction of δ^n, we have $\delta^n(\bar{c}) = \bar{a} = a + \operatorname{Im} d_A^n$, where $c = p^n(b)$, and $d_B^n(b) = i^{n+1}(a)$ for some $b \in B^n$ and $a \in A^{n+1}$. We have

$$\big(H^{n+1}(i^\cdot)\cdot\delta^n\big)(\bar{c}) = H^{n+1}(i^\cdot)(\bar{a}) = \overline{i^{n+1}(a)} = i^{n+1}(a) + \operatorname{Im} d_B^n =$$

$$= d_B^n(b) + \operatorname{Im} d_B^n = \operatorname{Im} d_B^n \text{ i.e. } \big(H^{n+1}(i^\cdot)\cdot\delta^n\big)(\bar{c}) = \bar{0} \text{ in } H^{n+1}(B).$$

Conversely let $\bar{a} \in \operatorname{Ker} H^{n+1}(i^\cdot)$ i.e. $\bar{a} = a + \operatorname{Im} d_A^n$ for some $a \in A^{n+1}$ with $d_A^{n+1}(a) = 0$ such that $0 = H^{n+1}(i^\cdot)(\bar{a}) = \overline{i^{n+1}(a)} = i^{n+1}(a) + \operatorname{Im} d_B^n$.

It follows $i^{n+1}(a) \in \operatorname{Im} d_B^n$ i.e. $i^{n+1}(a) = d_B^n(b)$ for some $b \in B^n$.

Denoting $c = p^n(b) \in C^n$, we observe that $d_C^n(c) = (d_C^n \cdot p^n)(b) =$

$$= (p^{n+1}\cdot d_B^n)(b) = p^{n+1}\big(d_B^n(b)\big) = \underbrace{(p^{n+1}\cdot i^{n+1})}_{=0}(a) = 0 \text{ so } \bar{c} = c + \operatorname{Im} d_C^{n-1} \in H^n(C^\cdot).$$

Moreover $\delta^n(\bar{c}) = \bar{a}$ by construction of δ^n, so $\operatorname{Ker} H^{n+1}(i^\cdot) \subseteq \operatorname{Im}\delta^n$ q.e.d.

Another way to compute cohomology is to change the complex with one which is simpler but with the same cohomology. In order to explain this, consider two ~~map~~ morphisms of complexes $f^\cdot, g^\cdot : A^\cdot \longrightarrow B^\cdot$. These morphisms are called <u>homotopic</u> if there exist a sequence of R-linear map $k = (k^n)_{n\in\mathbb{Z}}$, with $k^n : A^n \longrightarrow B^{n-1}$ such that $f^n - g^n = d_B^{n-1}\cdot k^n + k^{n+1}\cdot d_A^n$.

$$\begin{tikzcd}
A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, shift left] \arrow[d, shift right] & A^n \arrow[r, "d_A^n"] \arrow[dl, "k^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & A^{n+1} \arrow[dl, "k^{n+1}"'] \arrow[d, shift left] \arrow[d, shift right] \\
B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1}
\end{tikzcd}$$

<u>Exercise 5</u>: Verify that the homotopy relation is an equivalence, i.e. it is reflexive, transitive and symmetric.

A complex $A^\cdot$ is called <u>contractible</u> if the identity morphism $1_{A^\cdot} : A^\cdot \to A^\cdot$ is ~~homotopict~~ homotopic to the zero map. Two complexes $A^\cdot$ and $B^\cdot$ are called <u>homotopically equivalent</u> if there are morphisms of complexes $f^\cdot : A^\cdot \to B^\cdot$ and $g^\cdot : B^\cdot \to A^\cdot$ such that $g^\cdot f^\cdot \overset{h}{\sim} 1_{A^\cdot}$ and $f^\cdot g^\cdot \overset{h}{\sim} 1_{B^\cdot}$ where $\overset{h}{\sim}$ denotes the ~~relaton~~ relation "is homotopic to".

<u>Exercise 6</u> Verify that the relation "homotopically equivalent" is indeed an equivalence relation on complexes.

<u>Proposition</u> 2.2.5 If $f^\cdot, g^\cdot : A^\cdot \to B^\cdot$ are morphisms of complexes and $f^\cdot \overset{h}{\sim} g^\cdot$ then $H^n(f^\cdot) = H^n(g^\cdot)$ for all $n \in \mathbb{Z}$.

<u>Proof</u> ~~It suffices to verify that if~~ $\cancel{f^\cdot - g^\cdot}$ ~~is homotopic to the zero map, then~~ $\cancel{H^n(f^\cdot - g^\cdot) = 0}$. ~~But for every~~ $\cancel{\bar a \in H^n(A^\cdot)}$ ~~i.e.~~ $\cancel{\bar a = a + \operatorname{Im} d_A^{n-1}}$, ~~with~~ $\cancel{d_A^n(a) = 0}$, ~~we have~~ $\cancel{H^n(f^\cdot - g^\cdot)(\bar a) = \overline{(f^n - g^n)(a)} = \overline{f^n(a) - g^n(a)} = f^n(a) - g^n(a) + \operatorname{Im} d_B^{n-1}}$. ~~If~~ $\cancel{f^\cdot \overset{h}{\sim} g^\cdot}$ ~~via~~ $\cancel{k = (k^n)_{n \in \mathbb{Z}}, k^n : A^n \to B^{n-1}}$ ~~then~~

For every $\bar a \in H^n(A^\cdot)$ i.e. $\bar a = a + \operatorname{Im} d_A^{n-1}$ with $d_A^n(a) = 0$, we have
$$H^n(f^\cdot)(\bar a) = \overline{f^n(a)} = f^n(a) + \operatorname{Im} d_B^{n-1} \text{ and } H^n(g^\cdot)(\bar a) = g^n(a) + \operatorname{Im} d_B^{n-1}.$$
If $f^\cdot \overset{h}{\sim} g^\cdot$ via $k = (k^n)_{n \in \mathbb{Z}}$, $k^n : A^n \to B^{n-1}$, then
$$f^n(a) - g^n(a) = (f^n - g^n)(a) = (d_B^{n-1} \cdot k^n + k^{n+1} \cdot d_A^n)(a) = d_B^{n-1}(k^n(a)) + k^{n+1}(\underset{0}{\underbrace{d_A^n(a)}})$$
$$= d_B^{n-1}(k^n(a)) \in \operatorname{Im} d_B^{n-1}$$
so $f^n(a) + \operatorname{Im} d_B^{n-1} = g^n(a) + \operatorname{Im} d_B^{n-1}$, and $H^n(f^\cdot)(\bar a) = H^n(g^\cdot)(\bar a)$.

<u>Corollary</u> 2.2.6 a) If $A^\cdot$ is a contractible complex, then $H^n(A^\cdot) = 0$ for all n.

b) If $A^\cdot$ and $B^\cdot$ are homotopically equivalent complexes, then $H^n(A^\cdot) \cong H^n(B^\cdot)$ for all n.

<u>Remark</u> 2.2.7. a) There is a concept of homotopy in topology, as follows: Two continuous maps of topological spaces $\varphi, \psi : X \to Y$ are said to be homotopic ($\varphi \overset{h}{\sim} \psi$) if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = \varphi(x)$ and $F(0,x) = \psi(x)$ for all $x \in X$. The topological spaces X and Y are called homotopically equivalent if there are continuous maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \varphi \overset{h}{\sim} 1_X$, $\varphi \circ \psi \overset{h}{\sim} 1_Y$. Clearly two ~~topologi~~ homeomorph topo-

logical spaces are also homotopically equivalent.

b) Assume now that X and Y are triangulated spaces, and $\varphi, \psi : X \to Y$ are continuous maps. Then φ and ψ induce morphisms of complexes $\varphi_\bullet, \psi_\bullet : C_\bullet(X) \to C_\bullet(Y)$. Note that if $\varphi \overset{h}{\sim} \psi$ then $\varphi_\bullet \overset{h}{\sim} \psi_\bullet$. Consequently, if X and Y are homotopically equivalent, then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.

<u>Exercise 7</u> Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes. Define $\mathrm{Cone}(f)^\bullet = (\mathrm{Cone}(f)^n)_{n\in\mathbb{Z}}$, where ~~$\mathrm{Cone}(f)^n = A_{n-1} \oplus$~~

$$\mathrm{Cone}(f)^n = A^{n-1} \oplus B^n,$$

and with differentials

$$\cdots \mathrm{Cone}(f)^n \xrightarrow{d^n} \mathrm{Cone}(f)^{n+1} \longrightarrow \cdots, \quad d^n(a,b) = (d(a), -d(b)+f(b))$$

or equivalently $d^* = \begin{pmatrix} d_A & 0 \\ f & -d_B \end{pmatrix}$. Verify that what we obtain is indeed a complex, called the <u>mapping cone</u> of f.

<u>Exercise 8</u> The same ~~problem~~ question as in <u>exercise 7</u> for the <u>mapping cylinder</u> ~~i.e. the~~ of a morphism $f^\bullet : A^\bullet \to B^\bullet$, i.e. the complex

$$\mathrm{Cyl}(f)^\bullet = (\cdots \to \mathrm{Cyl}(f)^n \xrightarrow{d^n} \mathrm{Cyl}(f)^{n+1} \longrightarrow \cdots)$$

where $\mathrm{Cyl}(f)^n = A^n \oplus A^{n-1} \oplus B^n$ and $d = \begin{pmatrix} d_A & 1_A & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}$.

<u>Exercise 9</u> (The 3×3 lemma). Consider the following commutative diagrams with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A' \arrow[d] & B' \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & A'' \arrow[d] & B'' \arrow[d] & C'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

a) If in the first diagram the rows are also exact, then it may be completed to a commutative diagram with an exact row $0 \to A' \to B' \to C' \to 0$.

b) If in the second diagram the rows are also exact, then it may be completed to a commutative diagram with an exact row $0 \to A'' \to B'' \to C'' \to 0$.

<u>Exercise 10</u> If in the commutative diagram of R-modules and R-linear maps

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

columns and the first and the third row are exact and the composite $A \to B \to C$ is zero, then the second row is also exact.

<u>Exercise 11</u>. Consider the following commutative diagram of R-modules

$$\begin{tikzcd}
 & A' \arrow[r, "\alpha'"] \arrow[d, "f"'] & B' \arrow[r, "\beta'"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & {} &
\end{tikzcd}$$

(the <u>snake lemma</u>)

Then there is a map ~~f: coker~~ $\delta : \ker h \to \operatorname{coker} f$ making exact the sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

where the rest of homomorphisms are the induced ones.

<u>Exercise 12</u> (The five lemma). Consider the following diagram with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5
\end{tikzcd}$$

a) If f_1 is an epimorphism and f_2, f_4 are monomorphisms, then f_3 is a monomorphism.

b) If f_5 is a monomorphism and f_2, f_4 are epimorphisms, then f_3 is an epimorphism.

c) If f_1 is an epimorphism, f_5 is a monomorphism and f_2, f_4 are isomorphisms then f_3 is an isomorphism.