3.4. EXACTNESS OF FUNCTORS

All categories which we deal with in this section are abelian and all functors are additive. So when we speak about a functor we mean an additive one.

Let $\mathcal{C}$ and $\mathcal{D}$ two abelian categories. A functor $T : \mathcal{C} \to \mathcal{D}$ is called *left (right) exact* if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, the induced sequence

$$0 \to T(A) \to T(B) \to T(C) \text{ (respectively } T(A) \to T(B) \to T(C) \to 0)$$

is exact in $\mathcal{D}$. The functor T is *exact* provided that it is both left and right exact i.e. it carries the exact sequence above into an exact sequence

$$0 \to T(A) \to T(B) \to T(C) \to 0.$$

Remark that every (additive) functor preserves split exact sequences.

Lemma 3.4.1. *A functor $T : \mathcal{C} \to \mathcal{D}$ is exact if and only if it it sends every exact sequence in $\mathcal{C}$ into an exact sequence in $\mathcal{D}$.*

Proof. The sufficiency of the condition is obvious. For the necessity let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence in $\mathcal{C}$. It gives rise to three short exact sequences

$$0 \to \operatorname{Ker} \alpha \to A \to \operatorname{Im} \alpha \to 0$$

$$0 \to \operatorname{Ker} \beta \to B \to \operatorname{Im} \beta \to 0$$

$$0 \to \operatorname{Im} \beta \to C \to C/\operatorname{Im} \beta \to 0$$

with $\operatorname{Im} \alpha = \operatorname{Ker} \beta$. Since T preserves the exactness of each one of these sequences, it follows $\operatorname{Im} T(\alpha) = T(\operatorname{Im} \alpha)$ and $\operatorname{Ker} T(\beta) = T(\operatorname{Ker} \beta)$, therefore $\operatorname{Im} T(\alpha) = \operatorname{Ker} T(\beta)$ and the sequence $T(A) \xrightarrow{T(\alpha)} T(B) \xrightarrow{T(\beta)} T(C)$ is exact. $\square$

Exercise 1. Show that a functor $T : \mathcal{C} \to \mathcal{D}$ is left (right) exact if and only if it preserves the exactness of sequences of the form $0 \to A \to B \to C$ (respectively $A \to B \to C \to 0$).

A contravariant functor $T : \mathcal{C} \to \mathcal{D}$ is said to be left exact if the corresponding covariant functor $T : \mathcal{C}^{\text{op}} \to \mathcal{D}$ has this property, more exactly if it sends a short exact sequence $0 \to A \to B \to C \to 0$ (or, using Exercise 1 an exact sequence $A \to B \to C \to 0$) to an exact sequence $0 \to T(C) \to T(B) \to T(A)$. By duality we define a right exact contravariant functor.

Proposition 3.4.2. *Let $X \in \mathcal{C}$ be a fixed object of the abelian category $\mathcal{C}$. Then the functors*

$$\mathcal{C}(X,-) : \mathcal{C} \to \mathcal{A}b \text{ (covaraiant)}$$
$$\mathcal{C}(-,X) : \mathcal{C} \to \mathcal{A}b \text{ (contravaraiant)}$$

are left exact. (Compare with Theorem 1.2.3!).

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{C}$. Hence $A \cong \mathrm{Ker}(B \to C)$, so for every morphism $X \to B$ such that the composition $X \to B \to C$ vanishes, there is a unique morphism $X \to A$, such that $X \to B = X \to A \to B$. Thus $\mathcal{C}(X,A)$ is the kernel of the induced group homomorphism $\mathcal{C}(X,B) \to \mathcal{C}(X,C)$, showing that the sequence

$$0 \to \mathcal{C}(X,A) \to \mathcal{C}(X,B) \to \mathcal{C}(X,C)$$

is exact in $\mathcal{A}b$.

Exercise 2. Prove the left exactness of the contravariant Hom functor.

□

An object X of an abelian category $\mathcal{C}$ is called *projective* (*injective*) if the functor $\mathcal{C}(X,-)$ (respectively $\mathcal{C}(-,X)$) is exact. More explicitly an object $X \in \mathcal{C}$ is projective (injective) if and only of every diagram with exact row in $\mathcal{C}$:

$$\begin{tikzcd} & X \arrow[d] \arrow[dl, dotted] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \quad \left(\text{respectively} \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dotted] \\ & X & \end{tikzcd} \right)$$

may be completed commutative with the dotted arrow.

Lemma 3.4.3. *Let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{C}$. Then:*

a) *$\bigoplus_I X_i$ is projective if and only if each X_i is projective.*
b) *$\prod_I X_i$ is injective if and only if each X_i is injective.*

Proof. a) Denote $X = \bigoplus_I X_i$. For all $i \in I$ let denote $\rho_i : X_i \to X$ the canonical injection of the coproduct. Consider the diagram with exact row

$$\begin{tikzcd} X_i \arrow[r, "\rho_i"] \arrow[d, dotted, "\xi_i"'] & X \arrow[d] \arrow[dl, dotted, "\xi"] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

If every X_i is projective, then the diagram may be completed commutative with ξ_i for all $i \in I$, so it may be completed also with ξ which is induced by the definition of the coproduct, and X is projective as well. Conversely if X is projective, then the diagram may be completed commutative with ξ, and putting $\xi_i = \xi\rho_i$, we deduce that X is projective. The statement from b) follows by duality.

Exercise 3. Prove directly the point b) above.

□

Remark 3.4.4. An alternative proof of the previous Lemma uses the exactness of the direct product in $\mathcal{A}b$ and the formulae:

$$\mathcal{C}\left(\bigoplus_I X_i, -\right) \cong \prod_I \mathcal{C}(X_i, -) \text{ and } \mathcal{C}\left(-, \prod_I X_i\right) \cong \prod_I \mathcal{C}(-, X_i).$$

Let P be an R-module, where R is an arbitrary ring. A *dual basis* for P is a subset $\{x_i \mid i \in I\} \subseteq P$ together with R-linear maps $\varphi_i : P \to R$ for all $i \in I$ such that for each $x \in P$ we have $\varphi_i(x) = 0$ for almost all $i \in I$ and one has

$$x = \sum_{i \in I} \varphi_i(x) x_i.$$

Proposition 3.4.5. *The following are equivalent for an object $P \in \mathcal{C}$:*

(i) *P is projective.*
(ii) *Every exact sequence $0 \to A \to B \to P \to 0$ splits.*

*Moreover if $\mathcal{C} = R$-*Mod*, where R is an arbitrary ring, then the conditions (i) and (ii) are also equivalent to:*

(iii) *P is a direct summand of a free module.*
(iv) *P has a dual basis.*

Proof. (i)⇒(ii). If P is projective, then the diagram with exact row

$$\begin{tikzcd} & & & P \arrow[dl, dotted] \arrow[d, equal, "1_P"] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

may be completed commutative with the dotted arrow, so the row split by Proposition 3.3.3.

(ii)⇒(i). Consider an epimorphism $B \to C$ and complete it to an exact sequence $0 \to A \to B \to C \to 0$, by taking $A \to B = \ker(B \to C)$. For every morphism $P \to C$ construct the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where the right square is a pullback (see Section 3.3, Exercise 16). By hypothesis the upper row splits, so there is a morphism $P \to B'$ such that $P \to B' \to P$ is the identity on P. The composite morphism $P \to B' \to B$ has the property

$$P \to B' \to B \to C = P \to B' \to P \to C = P \to C,$$

showing that P is projective.

Suppose now that $\mathcal{C} = R$-Mod. Then $\mathrm{Hom}_R(R, M) \cong M$, so $\mathrm{Hom}_R(R, -)$ is exact, showing that R is projective. Using Lemma 3.4.3, every free module (that is a direct sum of copies of R) is also projective. Thus (ii)$\Rightarrow$(iii) is a consequence of the fact that every module is a quotient of a free module, whereas (iii)$\Rightarrow$(i) follows again from Lemma 3.4.3.

Now we want to prove the equivalence of the previous three conditions to (iv). For that let $\beta : F \to P$ be an epimorphism of a free module F into P. Let also $(e_i)_{i\in I}$ be a basis for F, so $F = R^{(I)}$. If P is projective then β splits, so there is $\varphi : P \to F$, such that $\beta\varphi = 1_P$. Composing it with the projections $F = R^{(I)} \to R$ for all $i \in I$, we obtain homomorphisms $\varphi_i : P \to R$, with $i \in I$. Thus the set with elements $x_i = \beta(e_i) \in P$ for all $i \in I$ is a dual basis. Conversely if $\{x_i \mid i \in I\} \subseteq P$ together with R-linear maps $\varphi_i : P \to R$ for all $i \in I$ is a dual basis for P then the R-linear maps φ_i with $i \in I$ induces, by the definition of the direct sum, an R-linear map $\varphi : P \to R^{(I)}$. Clearly $\beta\varphi = 1_P$, so P is a direct summand of F, so it is projective. $\square$

In the proof of the characterization of injective modules bellow we need the following set theoretic result, well–known as Zorn's Lemma. We shall not prove the Zorn's lemma, but we only mention that it is equivalent to the famous axiom of choice.

Lemma 3.4.6. *If $(A, \leq)$ is a poset (partially ordered set) such that every ascending chain $x_0 \leq x_1 \leq x_2 \leq \ldots$ in A has an upper bound, then for every $x \in A$ there is a maximal element $m \in A$ such that $x \leq m$.*

Proposition 3.4.7. *The following are equivalent for an object $E \in \mathcal{C}$:*

(i) *E is injective.*
(ii) *Every exact sequence $0 \to E \to B \to C \to 0$ splits.*

*Moreover if $\mathcal{C} = R$-*Mod*, where R is an arbitrary ring, then the conditions (i) and (ii) are also equivalent to:*

(iii) *For every left ideal I of R and every R-linear map $\varphi : I \to E$ there exists $x \in E$ such that $\varphi(a) = ax$ for all $a \in I$.*

Note that the condition (iii) above is called the *Baer's criterion of injectivity.*

Proof. The equivalence between (i) and (ii) follows from Proposition 3.4.5 by duality.

Exercise 4. Prove directly the equivalence between (i) and (ii) above.

Suppose now that $\mathcal{C} = R$-Mod, and E is an R-module. Recall that a left ideal I of R is nothing but a left R-submodule of R. Note that every R-linear $R \to E$ is of the form $a \mapsto ax$ for some $x \in E$ (actually x corresponds to $1 \in R$ under this map). Therefore the Baer's criterion follows as a particular case of the definition of injectivity, more precisely it says that $\mathrm{Hom}_R(-, E)$ preserves the exactness of the short exact sequence $0 \to I \to R \to R/I \to 0$.

Conversely, suppose that E satisfies the Baer's criterion of injectivity, let $\alpha : L \to M$ be a monomorphism (without to loss the generality, we may consider α as an inclusion of a submodule L into R-module M), and let $\varphi : L \to E$ be any R-linear map. Consider the set

$$\mathcal{M} = \{(L', \varphi') \mid L \leq L' \leq M \text{ and } \varphi' : L' \to E \text{ extends } \varphi\}.$$

Note that φ' extends φ means that there is a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\varphi"'] & L' \arrow[dl, "\varphi'"] \\ E & \end{tikzcd}$$

where $L \to L'$ is the inclusion. We want to apply Zorn's lemma in order to show that φ may be extended to a linear map $\psi : M \to E$, that is $(M, \psi) \in \mathcal{M}$. Observe first that $\mathcal{M}$ is ordered by declaring $(L', \varphi') \leq (L'', \varphi'')$ if $L' \leq L''$ and φ'' extends φ'. Moreover if $(L_0, \varphi_0) \leq (L_1, \varphi_1) \leq \dots$ is an ascending chain in $\mathcal{M}$, we construct $L_\infty = \bigcup_{i \geq 0} L_i$, and $\varphi_\infty : L_\infty \to E$ given as follows: for $x \in L_\infty$, we have $x \in L_n$ for some $n \geq 0$, and we put $\varphi_\infty(x) = \varphi_n(x)$.

Exercise 5. Verify that $(L_\infty, \varphi_\infty) \in \mathcal{M}$ and it is an upper bound of the chain $(L_0, \varphi_0) \leq (L_1, \varphi_1) \leq \dots$ considered above.

By Zorn's lemma there is a maximal element $(\bar{L}, \bar{\varphi})$, such that $(L, \varphi) \leq (\bar{L}, \bar{\varphi})$. Suppose that $\bar{L} < M$. Then there is $y \in M$ such that $y \notin \bar{L}$. Put $I = \{a \in R \mid ay \in \bar{L}\}$ which is a left ideal of R. The map $I \to E$, given by $a \mapsto \bar{\varphi}(ay)$ is R-linear, so by (iii) there is $x \in E$ such that $\bar{\varphi}(ay) = ax$ for all $a \in I$. Now put $L' = \bar{L} + Ry$ and define $\varphi' : L' \to E$ as $\varphi'(z + ay) = \bar{\varphi}(z) + ax$.

Exercise 6. Verify that the map φ' given above is well defined and it is R-linear. Moreover φ' extends $\bar{\varphi}$.

By the above Exercise, it is clear that $(\bar{L}, \bar{\varphi}) < (L'\varphi')$ in $\mathcal{M}$, contradicting the maximality of $(\bar{L}, \bar{\varphi})$. Therefore $\bar{L} = M$, so we put $\psi = \bar{\varphi}$ which is an R-linear map extending φ. $\square$

A poset $(I, \leq)$ is called *directed* if for any two elements $i, j \in I$ have an upper bound in I, that is there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Recall that every poset may be regarded as a small category satisfying the additional property that there is at most morphism between every two objects. A *direct system* in R-Mod is a functor $I \to R$-Mod, where I is a directed poset. Informally a direct system is a family $(M_i)_I$ of R-modules (indexed over I), together with R-linear maps $f_{ji} : M_i \to M_j$, for all $i \leq j \in I$, such that for every $i \leq j \leq k$ we have $f_{ki} = f_{kj} f_{ji}$. Given a directed system $(M_i, f_{ji})_I$ we take the disjoint union $\coprod_I M_i$ of sets M_i, and we define a binary relation $\sim$ on $\coprod_I M_i$ as follows: For $x, y \in \coprod_I M_i$, we have $x \in M_i$ and $y \in M_j$ for some $i, j \in I$ and we put $x \sim y$ if there is $k \in I$ such that $i \leq k$ and $j \leq k$ such that $f_{ki}(x) = f_{kj}(y)$.

Exercise 7. Show that the above defined relation $\sim$ is an equivalence relation on $\coprod_I M_i$ (provided that I is directed).

Now we put $\varinjlim M_i = \coprod_I M_i/\sim$. Roughly speaking this means that in $\varinjlim M_i$ we identify those elements which become equal for big $i \in I$. We observe also that there are (canonical) maps $f_i : M_i \to \varinjlim M_i$, sending each $x_i \in M_i$ into its equivalence class $[x_i]$, modulo the relation $\sim$. We have then clearly $f_j f_{ji} = f_i$ for all $i \leq j \in I$. Moreover every element $x \in \varinjlim M_i$ is of the form $x = [x_i] = f_i(x_i)$ for some $x_i \in M_i$; we say that x is represented by x_i. If $x, y \in \varinjlim M_i$, represented by $x_i \in M_i$, respectively $y_j \in M_j$ then choose $k \in I$ such that $i, j \leq k$ and define $x + y = [f_{ki}(x_i) + f_{kj}(y_j)]$, respectively $ax = [ax_i]$ for all $a \in R$.

Exercise 8. Show that the operations

$$\varinjlim M_i \times \varinjlim M_i \to \varinjlim M_i,\ (x, y) \mapsto x + y,$$

$$R \times \varinjlim M_i \to \varinjlim M_i,\ (a, x) \mapsto ax$$

given above are well defined (in the sense that their definitions do not depend on the choice of the representatives, and on the choice of k for the addition) and $\varinjlim M_i$ becomes an R-module relative to these operations. Further show that the maps $f_i : M_i \to \varinjlim M_i$ are R-linear.

The R-module $\varinjlim M_i$ together with R-linear maps $f_i : M_i \to \varinjlim M_i$, with $i \in I$, is called the *direct limit* of the system $(M_i, f_{ji})_I$.

Exercise 9. Show that the direct limit of a system $(M_i, f_{ji})_I$ satisfies and, up to a unique isomorphism, it is uniquely determined by, the following universal property: For every R-module M and every R-linear maps $g_i : M_i \to M$, with $i \in I$, for which $g_j f_{ji} = g_i$ for all $i \leq j \in I$, there is a unique R-linear map $g : \varinjlim M_i \to M$ such that $g f_i = g_i$ for all $i \in I$. Note that the universal property stated before may be visualized as the following commutative diagram:

$$\begin{tikzcd}
M_i \arrow[dd, "f_{ji}"'] \arrow[dr, "f_i"] \arrow[drr, "g_i"] & & \\
& \varinjlim M_i \arrow[r, dotted, "g"] & M \\
M_j \arrow[ur, "f_j"] \arrow[urr, "g_j"'] & &
\end{tikzcd}$$

Let fix now the directed set $(I \leq)$. Since a direct system of R-modules is a functor $I \to R$-Mod, a morphism of such direct systems must be a natural transformation between two such functors. Thus a morphism $(M_i, f_{ji})_I \xrightarrow{\varphi} (N_i, g_{ji})_I$ consists of R-linear maps $\varphi_i : M_i \to N_i$ for all $i \in I$ making commutative the diagram

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[d, "f_{ji}"'] & N_i \arrow[d, "g_{ji}"] \\ M_j \arrow[r, "\varphi_j"'] & N_j \end{tikzcd}$$

for all $i \leq j \in I$. We may consider then the category $(R\text{-Mod})^I$ of all direct systems of R-modules.

Exercise 10. Show that the direct limit

$$\varinjlim : (R\text{-Mod})^I \to R\text{-Mod}$$

gives rise to a functor.

An *exact sequence* of direct systems is a sequence

$$(L_i, f_{ji})_I \xrightarrow{\varphi} (M_i, g_{ji})_I \xrightarrow{\psi} (N_i, h_{ji})_I$$

of morphisms of such systems with the property that the sequence of R-modules $L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$ is exact for all $i \in I$.

Proposition 3.4.8. *The direct limit functor (see Exercise 10) is a exact.*

Proof. Let

$$0 \to (L_i, f_{ji})_I \xrightarrow{\varphi} (M_i, g_{ji})_I \xrightarrow{\psi} (N_i, h_{ji})_I \to 0$$

be a short exact sequence of direct systems, what means we have short exact sequences

$$0 \to L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i \to 0$$

for all $i \in I$. Denote $f_i : L_i \to \varinjlim L_i$, $g_i : M_i \to \varinjlim M_i$ and $h_i : N_i \to \varinjlim N_i$ the canonical morphisms of the direct limit. We have to prove that the direct limit sequence

$$0 \to \varinjlim L_i \xrightarrow{\bar{\varphi}} \varinjlim M_i \xrightarrow{\bar{\psi}} \varinjlim N_i \to 0$$

is exact in R-Mod, where $\bar{\varphi}$ and $\bar{\psi}$ comme from the universal property of the direct limit, being induced by the R-linear maps $\{g_i\varphi_i : L_i \to \varinjlim M_i \mid i \in I\}$, respectively $\{h_i\psi_i : M_i \to \varinjlim N_i \mid i \in I\}$. More precisely, if $x \in \varinjlim L_i$ is represented by $x_i \in L_i$ then $\bar{\varphi}(x) = [\varphi_i(x_i)]$, and similarly for $\bar{\psi}$. Note that we denote sometimes $\varinjlim \varphi_i = \bar{\varphi}$ and similarly $\varinjlim \psi_i = \bar{\psi}$.

First we show that φ is a monomorphism. Let $x \in \varinjlim L_i$ is represented by $x_i \in L_i$, such that $\bar{\varphi}(x) = 0$ in $\varinjlim M_i$. Thus $g_{ji}\varphi_i(x_i) = 0$ for some $j \geq i$. We have $\varphi_j f_{ji}(x_i) = g_{ji}\varphi_i(x_i) = 0$, and since φ_i is injective $f_{ji}(x_i) = 0$. Therefore $x = [x_i] = 0$ in $\varinjlim L_i$.

Now we want to show that $\operatorname{Ker}\bar{\psi} = \operatorname{Im}\bar{\varphi}$. Since $\bar{\psi}\bar{\varphi}f_i = h_i\psi_i\varphi_i = 0$, we deduce $\bar{\psi}\bar{\varphi} = 0$ from the uniqueness of the factorization through the direct limit, so $\operatorname{Im}\bar{\varphi} \leq \operatorname{Ker}\bar{\psi}$. Conversely let $y \in \operatorname{Ker}\bar{\psi} \leq \varinjlim M_i$ represented by $y_i \in M_i$. Thus $[\psi_i(y_i)]$ is zero in $\varinjlim N_i$, so $h_{ji}\psi_i(y_i) = 0$ for some $j \geq i$. If $y_j = g_{ji}(y_i)$ then $y = [y_i] = [y_j]$ and $\psi_j(y_j) = \psi_j g_{ji}(y_i) = h_{ji}\psi_i(y_i) = 0$, so $y_j \in \operatorname{Ker}\psi_j = \operatorname{Im}\varphi_j$. Thus $y_j = \varphi_j(x_j)$ for some $x_j \in L_j$, and $y = [y_j] = [\varphi_j(x_j)] = \bar{\varphi}([x_j]) \in \operatorname{Im}\bar{\varphi}$. Hence $\operatorname{Ker}\bar{\psi} \subseteq \operatorname{Im}\bar{\varphi}$.

Let $z \in \varinjlim N_i$ represented by $z_i \in N_i$. Since ψ_i is surjective, there is $y_i \in M_i$ such that $\psi_i(y_i) = z_i$. Then $\bar{\psi}([y_i]) = [z_i] = z$, showing that $\bar{\psi}$ is surjective.

Exercise 11. Without using elements (that is in a categorical way) show that $\bar{\psi}$ is an epimorphism.

$\square$

Fix a left R-module M. The Proposition 1.4.6 and the discussion preceding it show that the tensor product gives rise to an additive functor

$$- \otimes_R M : R\text{-Mod} \to \mathcal{A}b.$$

Proposition 3.4.9. *The tensor product functor commutes with direct limits, that is if $(L_i, f_{ji})_I$ is a direct system of right R-modules and M is a left R-module then there is a natural isomorphism*

$$\varinjlim (L_i \otimes_R M) \cong \left(\varinjlim L_i\right) \otimes_R M.$$

Proof. We denote by $f_i : L_i \to \varinjlim L_i$ the canonical morphisms of the direct limit. Since $(L_i \otimes_R M, f_{ji} \otimes_R M)$ is a direct system of abelian groups, the R-linear maps $f_i \otimes_R M : L_i \otimes_R M \to \left(\varinjlim L_i\right) \otimes_R M$ induce a unique abelian group homomorphism $f : \varinjlim (L_i \otimes_R M) \to \left(\varinjlim L_i\right) \otimes_R M$, such that $fg_i = f_i \otimes_R M$ for all $i \in I$, where $g_i : L_i \otimes_R M \to \varinjlim (L_i \otimes_R M)$ are the canonical morphisms of this direct limit of the above direct system of abelian groups. Note that $f([x_i \otimes y]) = [x_i] \otimes y$ for all $x_i \in L_i$ and all $y \in M$. On the other hand the map

$$\left(\varinjlim L_i\right) \times M \to \varinjlim (L_i \otimes_R M)\,,\ ([x_i], y) \mapsto [x_i \otimes y]$$

is bilinear, so it induces a unique abelian group homomorphism

$$g : \left(\varinjlim L_i\right) \otimes_R M \to \varinjlim (L_i \otimes_R M) \text{ such that } g([x_i] \otimes y) = [x_i \otimes y].$$

We observe immediately that g is the inverse of f, so f is an isomorphism. The proof ends with the help of the following:

Exercise 12. Let $(L'_i, f'_{ji})_I$ be another direct system of right R-modules and M' be another left R-module, such that there is a morphism of direct systems

$\alpha = (\alpha_i)_I : (L_i, f_{ji})_I \to (L'_i, f'_{ji})_I$ and an R-linear map $\beta : M \to M'$. If f' is constructed as above, show that the diagram

$$\begin{array}{ccc} \varinjlim (L_i \otimes_R M) & \xrightarrow{f} & \left(\varinjlim L_i\right) \otimes_R M \\ {\scriptstyle \varinjlim(\alpha_i\otimes\beta)}\downarrow & & \downarrow{\scriptstyle \left(\varinjlim \alpha_i\right)\otimes\beta} \\ \varinjlim (L'_i \otimes_R M') & \xrightarrow[f']{} & \left(\varinjlim L'_i\right) \otimes_R M' \end{array}$$

is commutative.

□

The left R-module F is called *flat* if the functor $- \otimes_R F$ is exact. Note that the tensor product is always right exact, so F is flat if and only if $f \otimes_R F$ is injective, whenever f is a monomorphism of right R-modules.

Proposition 3.4.10. *The following properties hold true:*

a) *A direct sum of modules is flat if and only if every term is flat.*
b) *A direct limit of flat modules is flat.*

Proof. a) Consider a family of left R-modules $(F_i)_I$, and a monomorphism of right R-modules $L \to M$. Since the tensor product of modules commutes with direct sums (see Proposition 1.4.4), we deduce that we have a commutative diagram

$$\begin{array}{ccc} L \otimes_R \left(\bigoplus_{i\in I} F_i\right) & \longrightarrow & M \otimes_R \left(\bigoplus_{i\in I} F_i\right) \\ \cong\downarrow & & \downarrow\cong \\ \bigoplus_{i\in I} (L \otimes_R F_i) & \longrightarrow & \bigoplus_{i\in I} (M \otimes_R F_i) \end{array}$$

The R-module $\bigoplus_{i\in I} F_i$ is flat if and only if the upper row is injective, or equivalently, the lower row is so, what means F_i is flat for all $i \in I$.

b) The conclusion follows as in the case of a) but using Proposition 3.4.9 instead of 1.4.4. □

Corollary 3.4.11. *Every projective module is flat.*

Proof. We have only to observe that R is flat as a left or right R-module, since $M \otimes_R R \cong M$. Hence the conclusion follows by Proposition 3.4.10. □

Exercise 13. An abelian group G is called *divisible* if for every $n \in \mathbb{N}^*$ and every $a \in G$ the equation $nx = a$ has at least a solution in G. Show that an abelian group is divisible if and only if it is injective. Consequently $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ (p is prime) and $\mathbb{Q}/\mathbb{Z}$ are injective abelian groups.

Exercise 14. Show that every vector space over a field K is both projective and injective.

Exercise 15. Find an example of a non free projective module. (Hint: Look at the direct summand of the ring $\mathbb{Z}(6) = \mathbb{Z}/6\mathbb{Z}$.)

Exercise 16. Let M be a module and let $(M_i, f_{ji})_I$ be a direct system of submodules, such that $f_{ij} : M_i \to M_j$ are inclusions. Then $\varinjlim M_i = \sum_I M_i$ and this sum coincide with the union of the sets M_i. (Compare with the union of the union $L_\infty = \bigcup_{i \geq 0} L_i$ in the proof of Proposition 3.4.7.)

Exercise 17. An abelian group G is called *torsion free* if every element $0 \neq x \in C$ has infinite order. Show that the following are equivalent for an abelian group G:

(i) G if torsion free.
(ii) For every $n \in \mathbb{N}^*$ and every $a \in G$ the equation $nx = a$ has at most a solution in G.
(iii) G is flat.

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