3.2. Kernels, cokernels, products and coproducts

In the sequel $\mathcal{C}$ will be a preadditive category.

A zero object in $\mathcal{C}$ is an object $Z \in \mathcal{C}$ such that ~~the~~ $\mathcal{C}(X,Z)=\{0\}$ and $\mathcal{C}(Z,X)=\{0\}$ for every $X \in \mathcal{C}$. Any two zero objects are isomorphic, therefore they will be identified to a single zero objects $0 \in \mathcal{C}$. If ~~the~~ a zero object 0 exists, then the neutral element of the group $\mathcal{C}(X',X)$ is identical to the composed morphism $X' \to 0 \to X$, as follows from the bilinearity of the composition of maps.

Example 3.2.1 The module with a single element $0=\{0\}$ is a zero object in the category R-Mod.

~~Exercise 1 Show that in an additive category $\mathcal{C}$, a mor~~

Exercise 1 Show that in a preadditive category $\mathcal{C}$, a morphism $\alpha : X' \to X$ is a monomorphism (resp. epimorphism) iff for every $\beta : X'' \to X'$ (resp. $\beta : X \to X''$) we have $\alpha\beta = 0$ (resp. $\beta\alpha=0$) implies $\beta = 0$.

Two monomorphisms $\alpha : X \to Y$ and $\alpha' : X' \to Y$ are called equivalent if there is an isomorphism $\gamma : X \to X'$ such that $\alpha'\gamma = \alpha$.

Exercise 2 Show that the relation defined above is actually an equiv. relation on the class of all monomorphisms with the same codomain $Y \in \mathcal{C}$.

An equivalence class of monomorphisms with the codomain Y is called a subobject of Y. By abuse of notation we will identify a ~~monomor.~~ subobject to a monomorphism representing it. If $X \xrightarrow{\alpha} Y$ and $X' \xrightarrow{\alpha'} Y$ are two subobjects of Y we write $X \leq X'$ if there is $\gamma : X \to X'$ such that $\alpha'\gamma = \alpha$.

Exercise 3 If $X \xrightarrow{\alpha} Y$ and $X' \xrightarrow{\alpha'} Y$ are subobjects and $X \leq X'$ then show that the morphism $\gamma : X \to X'$ for which $\alpha'\gamma = \alpha$ is a monomorphism. Show also that the relation "$\leq$" is reflexive and transitive on the class of all monomorphisms with the codomain Y. Consequently it induces an order on the class of all subobjects of Y (that is a relation which is reflexive, transitive and antisymmetric).

The dual notion of the notion of "subobject" is the so called quotient object. That is a quotient object is an equivalence class of epimorphisms with the same domain, when $X \xrightarrow{\beta} Y$, $X \xrightarrow{\beta'} Y'$ are equiv. if there exists an isomorphism $\gamma : Y \to Y'$ such that $\gamma\beta = \beta'$.

A kernel of a morphism $\alpha: X \to Y$ in $\mathcal{C}$ is a morphism $\kappa: K \to X$ such that $\alpha\kappa = 0$ and for every ~~$\xi: X \to Z$~~ $\xi: Z \to X$ such that $\alpha\xi = 0$ ~~there~~ is a unique $\gamma: Z \to K$ such that $\xi = \kappa\gamma$.

$$\begin{tikzcd} & Z \arrow[dl, dashed, "\gamma"'] \arrow[d, "\xi"] & \\ K \arrow[r, "\kappa"'] & X \arrow[r, "\alpha"] & Y \end{tikzcd}$$

Remark that the definition of a kernel is an universal property. Therefore we have:

Proposition 3.2.2 The kernel of a morphism is unique up to a unique isomorphism. Moreover the kernel of a morphism is a monomorphism, so any two kernels of $\alpha: X \to Y$ represent the same subobject of X.

Proof. The unicity of the kernel is standard: if $K \xrightarrow{\kappa} X$ and $K' \xrightarrow{\kappa'} X$ are two kernels of α then by definition of the kernel there are morphisms $\gamma: K' \to K$ and $\delta: K \to K'$ s.t. $\kappa' = \kappa\gamma$ and ~~$\kappa = \kappa\delta$~~ $\kappa = \kappa'\delta$

$$\begin{tikzcd} K' \arrow[dr, "\kappa'"] \arrow[dd, shift right, "\gamma"'] & & \\ & X \arrow[r, "\alpha"] & Y \\ K \arrow[ur, "\kappa"'] \arrow[uu, shift right, "\delta"'] & & \end{tikzcd}$$

Since $\kappa'\delta\gamma = \kappa\gamma = \kappa' = \kappa' 1_{K'}$ and

$\kappa\gamma\delta = \kappa'\delta = \kappa = \kappa 1_K$

we deduce by unicity stated in the definition of the kernel that $\delta\gamma = 1_{K'}$, $\gamma\delta = 1_K$ so γ is an isomorphism with $\delta = \gamma^{-1}$.

If $\kappa\gamma = 0 = \kappa 0$ for $\gamma: Z \to X$ it follows by unicity $\gamma = 0$, ~~for every $\gamma: Z \to X$.~~ So κ is a monomorphism and the above diagram shows that two kernels represent the same subobject of X.

Remark that because the unicity stated in Prop. 3.2.2 we may speak about the kernel of α if it exists. We denote the kernel by $\mathrm{Ker}\,\alpha$ and the inclusion by $\ker\alpha$(!).

Proposition 3.2.3 A morphism α is a monomorphism iff $\mathrm{Ker}\,\alpha = 0$.

Proof Exercise

The notion of cokernel is obtained from the one of the kernel by dualization. This means $\gamma: Y \to C$ is called the cokernel of $\alpha: X \to Y$ if $\gamma\alpha = 0$ and for every $\xi: Y \to Z$ such that $\xi\alpha = 0$ there is a unique $\varphi: C \to Z$ s.t. $\varphi\gamma = \xi$.

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \arrow[r, "\gamma"] \arrow[d, "\xi"] & C \arrow[dl, dashed, "\varphi"] \\ & Z & \end{tikzcd}$$

By dualization it follows that the cokernel is unique up to a unique (natural) isomorphism and two cokernels represent the same quotient object of Y.

Proposition 3.2.4 If α is the kernel of some morphism ~~then~~ and if $\mathrm{coker}\,\alpha$ exists ~~then~~ $\alpha = \ker(\mathrm{coker}\,\alpha)$.

Proof. Let $\alpha: X \to Y$ be the kernel of $\delta: Y \to Z$, and let $\gamma: Y \to C$ be the cokernel of α. Then $\delta\alpha = 0$ so there is a unique ~~morphism~~ $\varphi: C \to Z$ such that $\delta = \varphi\gamma$.

$$\begin{tikzcd} & U \arrow[dl, dashed, "\eta"'] \arrow[d, "\delta"] & \\ X \arrow[r, "\alpha"] & Y \arrow[r, "\gamma"] \arrow[d, "\xi"] & C \arrow[dl, dashed, "\psi"] \\ & Z & \end{tikzcd}$$

Let $\delta : U \to Y$ such that $\gamma\delta = 0$. It follows $\xi\delta = \psi\gamma\delta = 0$ and since $\alpha = \ker \xi$ we obtain a unique $\eta : U \to X$ s.t. $\alpha\eta = \delta$. This shows that $\alpha = \ker \gamma$.

Example 3.2.5 In the category R-Mod the kernel of an ~~R~~-linear map $\alpha : X \to Y$ is the ~~subset~~ submodule ~~$\mathrm{Ker}\,\alpha = \{a \in X \,/\, \alpha(a) = 0\}$~~

$$\mathrm{Ker}\,\alpha = \{x \in X \,/\, \alpha(x) = 0\} \quad \text{(together with the inclusion } \mathrm{Ker}\,\alpha \to X)$$

The cokernel of α is the projection $Y \longrightarrow Y / \mathrm{im}\,\alpha$.

Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of this family is an object $X \in \mathcal{C}$ together with morphisms $\pi_i : X \to X_i$ $(i \in I)$ such that for each object $Y \in \mathcal{C}$ and morphisms $\xi_i : Y \to X_i$ $(i \in I)$ there is a unique morphism $\xi : Y \to X$ such that $\pi_i \xi = \xi_i$ for all $i \in I$.

(i.e. the diagram is commut.:

$$\begin{tikzcd} X_i & X \arrow[l, "\pi_i"'] \\ & Y \arrow[ul, "\xi_i"] \arrow[u, "\xi"'] \end{tikzcd} \quad i \in I).$$

Exercise 5 Show that the product is unique up to a natural isomorphism.

We denote by $\prod_{i\in I} X_i$ the product of the family $(X_i)_{i\in I}$. The morphisms π_i are called the canonical projections.

For a family $\alpha_i : X_i \longrightarrow Y_i$ $(i \in I)$ of morphisms in a category with products $\mathcal{C}$, we define the product $\alpha = \prod_{i\in I} \alpha_i$ as being the unique morphism making commutative the diagram

$$\begin{tikzcd} X_i \arrow[r, "\alpha_i"] & Y_i \\ \prod X_i \arrow[u] \arrow[r, dashed, "\alpha"] & \prod Y_i \arrow[u] \end{tikzcd}$$

where the vertical morphisms are the canonical projections of the product. So the product may be regarded as a functor $\mathcal{C}^I \to \mathcal{C}$.

Proposition 3.2.6 If $\alpha_i : X_i \longrightarrow Y_i$ are monomorphisms, then so is $\alpha = \prod \alpha_i : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Y_i$ too.

Proof Let $\beta, \gamma : Z \to \prod_{i\in I} X_i$ such that $\alpha\beta = \alpha\gamma$. Denote by π_i^X and π_i^Y the canonical projections of the products $\prod X_i$ resp. $\prod Y_i$. We have then $\pi_i^Y \alpha\beta = \pi_i^Y \alpha\gamma$ so $\alpha_i \pi_i^X \beta = \alpha_i \pi_i^X \gamma$ for all $i \in I$. Since α_i is a monomorphism, we obtain $\pi_i^X \beta = \pi_i^X \gamma$ for all $i \in I$, and the unicity of the factorization through $\prod X_i$ implies $\beta = \gamma$.

Dualizing the notion of product we obtain the coproduct which is also called direct sum. Therefore an object $X \in \mathcal{C}$ together with a family of morphisms $s_i : X_i \longrightarrow X$ $(i \in J)$ is called the direct sum of the family of objects $X_i \in \mathcal{C}$ $(i \in J)$ if for every family of morphisms $\xi_i : X_i \longrightarrow Y$ there is a unique morphism $\xi : X \longrightarrow Y$ s.t. $\xi_i = \xi \circ s_i$, $(i \in J)$.

$$\begin{tikzcd} X_i \arrow[r, "s_i"] \arrow[dr, "\xi_i"'] & X \arrow[d, "\xi"] \\ & Y \end{tikzcd}$$

We denote $\bigoplus_{i \in J} X_i$ the direct sum and we call s_i canonical injections of the direct sum.

Exercise 6 Show that the canonical projections of the product are epimorphisms and the canonical injections of the direct sum are monomorph.

Exercise 7 Show that if $\{X_i \in \mathcal{C} \mid i \in J\}$ ~~with the pro~~ has the product in $\mathcal{C}$ then it holds $\mathcal{C}(X, \prod X_i) \cong \prod \mathcal{C}(X, X_i)$ for all $X \in \mathcal{C}$, where the second product is taken in the category $\mathcal{Ab}$ of all abelian groups. Conversely if $X \in \mathcal{C}$ is an object satisfying $\mathcal{C}(Y, X) \cong \prod_{i \in J} \mathcal{C}(Y, X_i)$ for all $Y \in \mathcal{C}$ then $X = \prod_{i \in J} X_i$. Dually the direct sum is characterized by the formula: $\mathcal{C}(\bigoplus X_i, X) \cong \prod \mathcal{C}(X_i, X)$ for all $X \in \mathcal{C}$.

~~Example 3.2.7 In category R-Mod, the products and~~ exist products ~~and direct sums for all family~~

Example 3.2.7 a) In category R-Mod, every family of objects (R-modules) has product and direct sum, as we have seen in Ch. 1, Section 3.
b) In the category Set, every family of objects has product and direct sum, namely the cartesian product resp. the disjoint union. (Exercise 8).

Note that, in general neither the product nor the direct sum of a family of objects in an arbitrary category ~~need exist~~ do need not exist. Our presentation is oriented to module categories where they are defined. ~~As~~ a first step, ~~as~~ the reader unfamiliar with the language of categories may see all our constructions in categories as another approach of modules, with the above ~~warning~~ warning that not all is valid for arbitrary categories.

Let $\{X_i \mid i\in I\}$ be a family of objects in $\mathcal{C}$ such that both $\prod X_i$ and $\bigoplus X_i$ exist. For any $i,j\in I$ we define the morphisms $\delta_{ij} : C_j \longrightarrow C_i$ as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \quad \text{(the Kronecker delta)}$$

From the definition of the product (direct sum), ~~then~~ for every $j\in I$ there is a unique morphism $\mathcal{f}_j' : X_j \longrightarrow \prod_{i\in I} X_i$ (resp. $\pi_j' : \bigoplus_{i\in I} X_i \longrightarrow X_j$) such that $\pi_i \mathcal{f}_j' = \delta_{ij}$ (resp. $\pi_j' \mathcal{f}_i = \delta_{ji}$) for all $i\in I$. (~~Here~~ we denoted π_i and $\mathcal{f}_i$ $(i\in I)$ the canonical projection resp. inj. of the product resp. direct sum.) In the case when I is finite, say $I=\{1,2,\dots,n\}$, we get

<u>Lemma</u> 3.2.8 a) $\mathcal{f}_1'\pi_1 + \dots + \mathcal{f}_n'\pi_n = 1_{\prod X_i}$

b) $\mathcal{f}_1\pi_1' + \dots + \mathcal{f}_n\pi_n' = 1_{\oplus X_i}$

(the notations are those used above).

<u>Proof</u> a) For all $i\in I$ we have $\pi_i(\mathcal{f}_1'\pi_1 + \dots + \mathcal{f}_n'\pi_n) = \delta_{i1}\pi_1 + \dots + \delta_{in}\pi_n =$ $= \pi_i = \pi_i \cdot 1$ so $\mathcal{f}_1'\pi_1 + \dots + \mathcal{f}_n'\pi_n = 1$ by the unicity of the factorization through the product.

b) <u>Exercise 9</u>

Conversely we have:

<u>Proposition</u> 3.2.9 Suppose there are given morphisms $\mathcal{f}_i : X_i \longrightarrow X$ and $\pi_i : X \longrightarrow X_i$, with $1\le i\le n$, such that

$$\pi_i \mathcal{f}_j = \delta_{ij} \ (1\le i,j\le n) \quad\text{and}\quad \mathcal{f}_1\pi_1 + \dots + \mathcal{f}_n\pi_n = 1_X$$

Then X is the product and the coproduct of objects $(X_i)_{i=\overline{1,n}}$ with the projections π_i and the injections $\mathcal{f}_i$.

<u>Proof</u> First we shall show that X is the coproduct. Let $\xi_i : X_i \longrightarrow Y$, $1\le i\le n$. Define $\xi = \xi_1\pi_1 + \dots + \xi_n\pi_n : X \longrightarrow Y$. Then $\xi \mathcal{f}_i = (\xi_1\pi_1 + \dots + \xi_n\pi_n)\mathcal{f}_i = \xi_1\delta_{1i} + \dots + \xi_n\delta_{ni} = \xi_i$ $(1\le i\le n)$. Moreover, if $\zeta : X\longrightarrow Y$ such that $\zeta\mathcal{f}_i = \xi_i$ $(1\le i\le n)$ then $\zeta = \zeta\cdot 1_X =$ $= \zeta(\mathcal{f}_1\pi_1 + \dots + \mathcal{f}_n\pi_n) = \zeta\mathcal{f}_1\pi_1 + \dots + \zeta\mathcal{f}_n\pi_n = \xi_1\pi_1 + \dots + \xi_n\pi_n = \xi$ so ξ is unique. Thus $X = \bigoplus_{i=1}^{n} X_i$. To show that $X = \prod_{i=1}^{n} X_i$ is <u>Exercise 10</u>.

The above result tells us that ~~is no need to distinguish between~~ products and direct sums over finite index sets coincide. We use notation $X_1\times X_2\times\dots\times X_n$ or $X_1\oplus X_2\oplus\dots\oplus X_n$.

It is convenient to use the matrix notation for morphisms between finite (co)products. A morphism $\alpha : \bigoplus_{j\in J} X_j \longrightarrow \bigoplus_{i\in I} Y_i$ will be identified to the matrix (α_{ij}) where $\alpha_{ij} = \pi_i^Y \alpha \, \rho_j^X : X_j \longrightarrow Y_i$, the composition of the morphisms being then the multiplication of matrices.

<u>Proposition</u> 3.2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories. A functor $T : \mathcal{C} \to \mathcal{D}$ is additive if and only if it preserves finite (co)products.

<u>Proof</u> If T is additive then it sends equalities characterizing finite (co)products from Prop. 3.2.9 into similar relations, so T preserves such (co)products. Conversely let $\alpha, \beta : X \to Y$ in $\mathcal{C}$. ~~The~~ and suppose that T preserves finite (co)products. Then $T(\alpha+\beta) = T(\alpha) + T(\beta)$ as we may see from the writing of $\alpha+\beta$ as the composition

$$X \xrightarrow{\binom{1}{1}} X \oplus X \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} Y \oplus Y \xrightarrow{(1,1)} Y .$$

<u>Exercice</u> 11. Consider the full subcategory Tors of the category $\mathcal{Ab}$ of all torsion groups. Recall that an abelian group G is torsion if every element of G is of finite order. Show that if $X_i \in$ Tors, $i \in I$ then $\bigoplus_{i\in I} X_i \in$ Tors ~~(here $\bigoplus_{i\in I}$~~ (here $\oplus$ means the direct sum in $\mathcal{Ab}$) and it is also a direct sum in Tors of the family $(X_i)_{i\in I}$. Give an contraexample, showing that generally $\prod_{i\in I} X_i \notin$ Tors. However Tors has products, the product of a family $X_i \in$ Tors $(i \in I)$ being constructed as the torsion part of $\prod_{i\in I} X_i$ (here $\prod$ means the products in $\mathcal{Ab}$). Recall that for an abelian group G the torsion part is $T(G) = \{ x \in G \mid \mathrm{ord}(x) \text{ is finite} \}$.

<u>Exercice</u> 12 ~~Show that~~ Let R be a commutative ring. Show that in the category of commutative R-algebras the coproduct of A and B is $A \otimes_R B$.

<u>Exercice</u> 13 Let $\alpha : X \to Y$ be a morphism in a preadditive category $\mathcal{C}$, such that α has kernel and there is $\beta : Y \to X$ such that $\alpha\beta = 1_Y$. Show that $X \cong Y \oplus \operatorname{Ker} \alpha$.