Chapter 3: Categories and Functors

3.1. Generalities about categories

Sets and classes – Gödel, Bernays, von Neumann axiomatisation of the set theory.

A *category* $\mathscr{C}$ is defined to consist of three ingredients:

1) a class $\mathrm{Ob}\,\mathscr{C}$ of objects of $\mathscr{C}$.
2) for each ordered pair (X', X) of objects, a set $\mathscr{C}(X', X)$ or $\mathrm{Hom}_{\mathscr{C}}(X', X)$, whose elements are called *morphisms* or *maps* from X' to X. We denote $f: X' \to X$ or $X' \xrightarrow{f} X$ an element $f \in \mathscr{C}(X', X)$.
3) for each ordered triple of objects (X', X, X'') a composition
$$\mathscr{C}(X, X'') \times \mathscr{C}(X', X) \longrightarrow \mathscr{C}(X', X''), \quad (\beta, \alpha) \mapsto \beta\alpha$$
(that means
$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[dr, "\beta\alpha"'] & X \arrow[d, "\beta"] \\ & X'' \end{tikzcd}$$
is commutative).

These data are the subject of the following axioms:

C1) $\mathscr{C}(X', X) \cap \mathscr{C}(Y', Y) = \emptyset$, provided that $(X', X) \neq (Y', Y)$.

C2) If $X_1 \xrightarrow{\alpha} X_2 \xrightarrow{\beta} X_3 \xrightarrow{\gamma} X_4$ are morphisms then
$$\gamma(\beta\alpha) = (\gamma\beta)\alpha$$
(when defined, the composition is associative)

C3) For each object X, there exists $1_X \in \mathscr{C}(X, X)$ such that $1_X\alpha = \alpha$ and $\beta 1_X = \beta$ for all $\alpha \in \mathscr{C}(X', X)$ and all $\beta \in \mathscr{C}(X, X'')$.

Exercise 1 Show that the morphism 1_X called also the *identity* of X is uniquely determined by X.

We write sometimes $X \in \mathscr{C}$ instead $X \in \mathrm{Ob}\,\mathscr{C}$, if no danger of confusion arroses.

Examples 3.1.1. a) The category Set: the class $\mathrm{Ob}(\mathcal{S}et)$ is the class of all sets, the morphisms are maps between sets with the usual composition

b) The category $\mathcal{A}b$: the class $\mathrm{Ob}(\mathcal{A}b)$ is the class of all abelian groups, the morphisms are homomorphisms of abelian groups with the usual composition.

c) The categories R-Mod and Mod-R, when R is a ring. The objects are left, resp. right R-modules, the maps are R-linear maps with the usual composition.

d) Every poset $(P, \leq)$ defines a category with P as the class of objects and with a unique morphism $p \to q$ if $p \leq q$, w… $\mathrm{Hom}(p, q) = \emptyset$ if $p \not\leq q$. This category is a small category, which means that the class of objects is actually a set.

For each category $\mathcal{C}$, we define the dual category $\mathcal{C}^{op}$ with $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}^{op}(X', X) = \mathcal{C}(X, X')$ for all $X, X' \in \mathrm{Ob}\,\mathcal{C}$ and for $\alpha: X'' \to X$, $\beta: X \to X'$ in $\mathcal{C}$ we have $\alpha * \beta = \beta\alpha$ (where $*$ denotes the composition in $\mathcal{C}^{op}$). So $\mathcal{C}^{op}$ is obtained from $\mathcal{C}$ by reversing the arrows. It is easily verified that $\mathcal{C}^{op}$ satisfies the axioms defining a category (Exercise 2). Every definition or theorem for $\mathcal{C}$ dualizes to a corresponding definition or theorem in $\mathcal{C}^{op}$.

~~A morphism $\alpha: X_1 \to X_2$ in a category $\mathcal{C}$ is an isomorphism (resp. monomorphism or epimorphism) if there is~~

A morphism $\alpha: X_1 \to X_2$ in a category $\mathcal{C}$ is an isomorphism if there is $\beta: X_2 \to X_1$ such that $\beta\alpha = 1_{X_1}$, $\alpha\beta = 1_{X_2}$ (we write $X_1 \cong X_2$). A morphism $\alpha: X_1 \to X_2$ is called a monomorphism (resp. epimorphism) if for every pair of morphisms $X \overset{\beta}{\underset{\gamma}{\rightrightarrows}} X_1$ (resp. $X_2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} X$) we have $\alpha\beta = \alpha\gamma \Rightarrow \beta = \gamma$ (resp. $\beta\alpha = \gamma\alpha \Rightarrow \beta = \gamma$).

Exercise 3 Show that an R-linear map is a monomorphism (resp. epimorphism) in R-Mod (in the categorical sense ~~def~~ above) iff it is injective (resp. surjective).

Exercise 4 Show that every isomorphism is both a monomorphism and an epimorphism, in every category $\mathcal{C}$. The converse is also true in Set, Ab or R-Mod (Mod-R). But generally? (Hint: the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both ~~a~~ monomorphism and an epimorphism in the category of all rings with identity, the morphisms being the unit preserving ring homomorphisms, but it is not an isomorphism).

~~We denote $X \cong X'$ if X and~~
~~(We write $X_1 \cong X_2$ for isomorphism)~~

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A <u>covariant</u> (resp. <u>contravariant</u>) <u>functor</u> $T: \mathscr{C} \to \mathscr{D}$ consists from the maps $\operatorname{Ob} \mathscr{C} \to \operatorname{Ob} \mathscr{D}$, $X \mapsto T(X)$ and $\mathscr{C}(X', X) \to \mathscr{D}(T(X'), T(X))$ (resp. $\mathscr{C}(X', X) \to \mathscr{D}(T(X), T(X'))$) $\alpha \mapsto T(\alpha)$ for each pair of objects (X', X) in $\mathscr{C}$, satisfying the axioms:

F1) $T(\beta\alpha) = T(\beta)\,T(\alpha)$ (resp. $T(\beta\alpha) = T(\alpha)\,T(\beta)$) for $X' \xrightarrow{\alpha} X \xrightarrow{\beta} X''$ morph.'s in $\mathscr{C}$.

F2) $T(1_X) = 1_{T(X)}$ for $X \in \mathscr{C}$.

If we speak about a functor we understand a covariant ones. Note also that a contravariant functor ~~may be viewed as a co~~ $T: \mathscr{C} \to \mathscr{D}$ may be viewed as a covariant functor $T: \mathscr{C}^{op} \to \mathscr{D}$. The functor $T: \mathscr{C} \to \mathscr{D}$ is called <u>faithful</u> (resp. <u>full</u>) if all the ~~maps~~ induced maps
$\mathscr{C}(X', X) \to \mathscr{D}(T(X'), T(X))$ are injective (resp. surjective). Functors $T: \mathscr{C} \to \mathscr{D}$ and $T': \mathscr{D} \to \mathscr{E}$ may in an obvious way be composed to a functor $T' \cdot T: \mathscr{C} \to \mathscr{E}$ by putting $(T' \cdot T)(X) = T'(T(X))$ and $(T' \cdot T)(\alpha) = T'(T(\alpha))$ for $X \in \operatorname{Ob} \mathscr{C}$ and $X' \xrightarrow{\alpha} X$ in $\mathscr{C}$.

Let $S, T: \mathscr{C} \to \mathscr{D}$ be functors. A <u>natural transformation</u> (or a <u>functorial homomorphism</u>) ~~is an assignment $\operatorname{Ob}\mathscr{C} \to \mathscr{D}$~~ $\eta: S \to T$ is an assignment ~~$\operatorname{Ob}\mathscr{C} \to \mathscr{D}$ S~~ $X \mapsto \eta_X$ for each $X \in \operatorname{Ob} \mathscr{C}$, where $\eta_X \in \mathscr{D}(S(X), T(X))$, so that for every morphism $\alpha: X' \to X$ in $\mathscr{C}$ one gets a commutative diagram

$$\begin{array}{ccc} S(X') & \xrightarrow{\eta_{X'}} & T(X') \\ {\scriptstyle S(\alpha)}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ S(X) & \xrightarrow{\eta_X} & T(X) \end{array} \qquad \left(\text{i.e. } T(\alpha)\,\eta_{X'} = \eta_X \cdot S(\alpha)\right)$$

A category $\mathscr{C}$ is <u>preadditive</u> if for each pair (X', X) of objects of $\mathscr{C}$, ~~the set $\mathscr{C}(X', X)$ has~~ there is an addition on the set $\mathscr{C}(X', X)$ making it into an abelian group such that the composition
$\mathscr{C}(X, X'') \times \mathscr{C}(X', X) \to \mathscr{C}(X', X'')$ is bilinear, what means
$\beta(\alpha_1 + \alpha_2) = \beta\alpha_1 + \beta\alpha_2$ and $(\beta_1 + \beta_2)\alpha = \beta_1\alpha + \beta_2\alpha$ for all $\beta, \beta_1, \beta_2 \in \mathscr{C}(X, X'')$ and all $\alpha, \alpha_1, \alpha_2 \in \mathscr{C}(X', X)$. If $\mathscr{C}, \mathscr{D}$ are preadditive categories, a functor $T: \mathscr{C} \to \mathscr{D}$ is called <u>additive</u> if the induced maps

$\mathcal{C}(X',X) \longrightarrow \mathcal{D}(T(X'),T(X))$ are group homomorphisms for every pair of objects (X',X) of $\mathcal{C}$, thus if it satisfies:

F3) $T(\alpha+\alpha') = T(\alpha) + T(\alpha')$ for all $\alpha,\alpha' : X' \to X$.

A <u>subcategory</u> of a category $\mathcal{C}$ is a category $\mathcal{B}$ such that $\mathrm{Ob}\,\mathcal{B} \subseteq \mathrm{Ob}\,\mathcal{C}$ (that means it is a subclass) and $\mathcal{B}(X',X)$ is a subset of $\mathcal{C}(X',X)$ for all $X',X \in \mathcal{B}$ and the composition in $\mathcal{B}$ is the same as the composition in $\mathcal{C}$. In that case, the inclusions give rise to a faithful functor $\mathcal{B} \longrightarrow \mathcal{C}$, called the <u>inclusion functor</u>. The subcategory $\mathcal{B}$ is called <u>full subcategory</u>, if this functor is full. If $\mathcal{C}$ is a preadditive ~~subcategory of~~ category and $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, then $\mathcal{B}$ is also preadditive and the inclusion functor is additive.

Let now I be a set and $\mathcal{C}_i$, $i\in I$ be a family of categories. The <u>product category</u> $\prod_{i\in I}\mathcal{C}_i$ is defined as

$$\mathrm{Ob}\Big(\prod_{i\in I}\mathcal{C}_i\Big) = \prod_{i\in I}\mathrm{Ob}\,\mathcal{C}_i \quad \text{and} \quad \mathrm{Hom}\big((X_i)_{i\in I},(Y_i)_{i\in I}\big) = \prod_{i\in I}\mathrm{Hom}(X_i,Y_i),$$

with composition defined component-wise.

<u>Example</u> 3.1.2 Let $\mathcal{C}$ be a category and fix $C\in\mathcal{C}$. Then the assignments $X \mapsto \mathcal{C}(C,X)$ and $\alpha \mapsto \alpha^* = \mathcal{C}(C,\alpha)$ for all $X\in\mathcal{C}$ and all $\alpha : X' \longrightarrow X$ in $\mathcal{C}$, where $\alpha^* : \mathcal{C}(C,X') \longrightarrow \mathcal{C}(C,X)$, $\alpha^*(f) = \alpha\cdot f$ respectively $X \mapsto \mathcal{C}(X,C)$ and $\alpha \mapsto \alpha_* = \mathcal{C}(\alpha,C)$ for all $X\in\mathcal{C}$ and all $\alpha : X' \longrightarrow X$ in $\mathcal{C}$, where $\alpha_* : \mathcal{C}(X,C) \longrightarrow \mathcal{C}(X',C)$, $\alpha_*(f) = f\cdot\alpha$ define two functors one of them covariant

$$\mathcal{C}(C,-) : \mathcal{C} \longrightarrow \mathrm{Set}$$

and the second contravariant

$$\mathcal{C}(-,C) : \mathcal{C} \longrightarrow \mathrm{Set}$$

(The verification that we indeed obtain functors are Exercice 5)
Then functors are called the covariant, respectively contravariant Hom-functor. We may regard them together as a <u>bifunctor</u>:

$$\mathcal{C}(-,-) : \mathcal{C}^{op}\times\mathcal{C} \longrightarrow \mathrm{Set}, \quad (X,Y) \mapsto \mathcal{C}(X,Y) \quad \text{and}$$

for $\alpha : X' \to X$, $\beta : Y' \to Y$ we have $\mathcal{C}(\alpha,\beta) : \mathcal{C}(X,Y') \to \mathcal{C}(X',Y)$

$$\mathcal{C}(\alpha,\beta)(f) = \alpha\cdot f\cdot\beta.$$

Exercise 6 If $\mathcal{C}$ is a preadditive category, and $C \in \mathcal{C}$ is fixed, show that the functors $\mathcal{C}(C,-)$ and $\mathcal{C}(-,C)$ are additive i.e. they may be regarded as functors $\mathcal{C}(C,-): \mathcal{C} \to Ab$ (covariant) respectively $\mathcal{C}(-,C): \mathcal{C} \to Ab$ (contravariant).

A natural transformation $\eta: S \to T$, when $S, T: \mathcal{C} \to \mathcal{D}$ are functors is called a natural equivalence, if $\eta_X : S(X) \to T(X)$ is an isomorphism in $\mathcal{D}$ for all $X \in \mathcal{C}$. In that case the functors S and T are called naturally equivalent, and we denote $S \cong T$.

Exercise 7 Show that the relation "naturally equivalent" is an equivalence relation on the class of all functors $\mathcal{C} \to \mathcal{D}$, that is it is reflexive, symmetric and transitive.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there exists functors $S: \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{D} \to \mathcal{C}$ such that $T \cdot S = 1_{\mathcal{C}}$ and $S \cdot T = 1_{\mathcal{D}}$. In practice this notion turns out to be too restrictive. In fact $\mathcal{C}$ and $\mathcal{D}$ have ~~the~~ essentially the same structure provided that they are equivalent, what is defined as: there are functors $S: \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{D} \to \mathcal{C}$ such that $T \cdot S \cong 1_{\mathcal{C}}$ and $S \cdot T \cong 1_{\mathcal{D}}$. In this case the functors ~~Proposition 3.1.3 A functor~~ S and T are called (mutually inverse) equivalences of categories.

Proposition 3.1.3 A functor $S: \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and every object $Y \in \mathcal{D}$ is isomorphic to an object of the form $S(X)$ for some $X \in \mathcal{C}$.

Proof. If S is an equivalence of categories and $T: \mathcal{D} \to \mathcal{C}$ such that $T \cdot S \cong 1_{\mathcal{C}}$, $S \cdot T \cong 1_{\mathcal{D}}$. If $\alpha, \alpha': X' \to X$ are morphisms in $\mathcal{C}$ such that $S(\alpha) = S(\alpha')$ then $(T \cdot S)(\alpha) = (T \cdot S)(\alpha')$ and the commutative diagrams

$$\begin{array}{ccc} X' & \xrightarrow{\cong} & (T\cdot S)(X') \\ \alpha \downarrow & & \downarrow (T\cdot S)(\alpha) \\ X & \xrightarrow{\cong} & (T\cdot S)(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} X' & \xrightarrow{\cong} & (T\cdot S)(X') \\ \alpha' \downarrow & & \downarrow (T\cdot S)(\alpha') \\ X & \xrightarrow{\cong} & (T\cdot S)(X) \end{array}$$

show that $\alpha = \alpha'$, so S is faithful. By symmetry T is also faithful. Let now $\beta: S(X') \to S(X)$ be a morphism in $\mathcal{D}$, where $X', X \in \mathcal{C}$. ~~The commuta~~ By the diagram:

$$\begin{tikzcd} X' \arrow[r, "\cong"] \arrow[d, dashed, "\alpha"'] & (T\cdot S)(X') \arrow[d, "T(\beta)"] \\ X \arrow[r, "\cong"] & (T\cdot S)(X) \end{tikzcd}$$

we get a morphism $\alpha : X' \to X$ such that $(T\cdot S)(\alpha) = T(\beta)$. Since T ~~is faithful~~ is faithful, we obtain $\beta = S(\alpha)$, so S is full. Finally for $Y \in \mathcal{D}$ we put $X = T(Y) \in \mathcal{C}$ and we have $Y \cong (S\cdot T)(Y) = S(X)$.

Conversely suppose that S is fully faithful and for every $Y \in \mathcal{D}$ there is $X \in \mathcal{C}$ such that $Y \cong S(X)$. Denote by $T(Y) = X$ and $\xi_Y : (S\cdot T)(Y) \to Y$ the isomorphism. ~~Since S is fully faithful, T is defined up to a natural isomorphism, i.e. if $X, X' \in \mathcal{C}$ such that $X \cong$~~ For a morphism $\beta : Y' \to Y$ in $\mathcal{D}$ we obtain a morphism $\xi_Y^{-1}\beta\,\xi_{Y'} : (S\cdot T)(Y') \to (S\cdot T)(Y)$ in $\mathcal{D}$ and since S is fully faithful, there is a unique ~~$\beta : T$~~ $\alpha : T(Y') \to T(Y)$ such that $\xi_Y^{-1}\cdot\xi\cdot\xi_{Y'} = S(\alpha)$. We put $T(\beta) = \alpha$. Verifications that what we do obtain is a functor $T : \mathcal{D} \to \mathcal{C}$ are <u>Exercise 8</u>.

Clearly $\xi : (S\cdot T) \to 1_{\mathcal{D}}$ is a natural equivalence. We want now to define a natural equivalence $\vartheta : T\cdot S \to 1_{\mathcal{C}}$. ~~Let $X \in \mathcal{C}$~~. For every $X \in \mathcal{C}$ we have an isomorphism $\xi_{S(X)} : (S\cdot T\cdot S)(X) \to S(X)$ and since S is fully faithful, there is a unique $\vartheta_X : (T\cdot S)(X) \to X$ such that $S(\vartheta_X) = \xi_{S(X)}$. The fact that ϑ_X is an isomorphism is <u>Exercise 9</u> and the naturality of ϑ is <u>Exercise 10</u>.

<u>Exercise 11</u>. Let R be a ring and let X be a set. Let $F_R(X)$ be the free R-module on the set X. By the universal property of free module (see Exercise 12, Chap. 1, Sect. 3), ~~we~~ every map of sets $\sigma : X' \to X$ induce a unique R-linear map $F_A(X') \to F_A(X)$; denote it with $F(\sigma)$. Show that we constructed a functor $F_A : Set \to R\text{-}Mod$. Let $U : R\text{-}Mod \to Set$, $U(M) = M$ and $U(f) = f$ the functor ~~whi which~~ which forgets the module structure. Show that the embeddings of the basis X in the ~~module~~ free module $\zeta_X : X \to (U\cdot F_R)(X)$ and the canonical map $\xi_M : (F_R\cdot G)(M) \to M$ from the free module on the set M to the R-module M are natural transformations.