I. 1. (*a*) Define and give an example of: ring, linear map, equivalence relation. **(9p)**
(*b*) State the Kronecker-Capelli theorem. **(4p)**
(*c*) Prove that n vectors in a K-vector space V are linearly dependent if and only if one of them can be written as a linear combination of the others. **(7p)**

2. Define the kernel and the image of a linear map and prove that they are subspaces in the domain, respectively codomain of that map. **(10p)**

3. Determine the rank of the matrix $A = \begin{pmatrix} 2 & 0 & 1 & -1 & 3 \\ 4 & 1 & 9 & -2 & 10 \\ 0 & 1 & 7 & 5 & -1 \\ 6 & 0 & 3 & -3 & \alpha \end{pmatrix}$ by using elementary operations. Discussion on $\alpha \in \mathbb{R}$. **(10p)**

4. (a) Define diagonalizable endomorphisms of a vector space. Give an example of diagonalizable endomorphism of ${}_{\mathbb{R}}\mathbb{R}^2$. State the theorem characterizing the diagonalizable endomorphisms. **(6p)**
(b) Let $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-2y - 3z, x + 3y + 3z, z)$. Write the matrix of f in the canonical basis $e = (e_1, e_2, e_3)$ of $\mathbb{R}^3$. **(4p)**

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II. 1. (*a*) Define and give an example of: group, basis, kernel of a linear map. **(9p)**

(*b*) State the theorem which relates the dimension of a sum of two subspaces of the dimension of their intersection. **(4p)**

(*c*) Let u, v, w be bases of the K-vector spaces U, V, respectively W and let $f : U \to V$ and $g : V \to W$ be linear maps. Show that $[g \circ f]_{u,w} = [g]_{v,w}[f]_{u,v}$. **(7p)**

2. State the characterization theorem of subspaces, and show that the intersection of a family of subspaces is a subspace. **(10p)**

3. (a) Solve the system $\begin{cases} x + 4y + 2z = 1 \\ 2x + 3y + z = 0 \\ 3x + \quad\ - z = \lambda \end{cases}$ by the Gauss method. Discussion on $\lambda \in \mathbb{R}$. **(10p)**

4. (a) Define the notion of eigenvalue of an endomorphism of a vector space. Give an example of an endomorphism of ${}_{\mathbb{R}}\mathbb{R}^2$ that has the eigenvalues 2 and 3. State the theorem that gives a practical method to compute eigenvalues. **(6p)**

(b) Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (x - y, x + y - 2z, -5y + z)$. Write the matrix of f in the canonical basis $e = (e_1, e_2, e_3)$ of $\mathbb{R}^3$. **(4p)**

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