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Remarks on triples in enriched categories

H. Wiesler and G. Calugareanu jr

Let V be a symmetric monoidal closed category with equalizers. The V-triples $\mathsf{T}, \mathsf{T}', \ldots$ in the enriched category A, together with suitably defined morphisms form a category V-**Trip**(A). The V-categories $A^{\mathsf{T}}, A^{\mathsf{T}'}, \ldots$ and the V-functors $R : A^{\mathsf{T}'} \to A^{\mathsf{T}}$ which are compatible with the forgetful functors form a category V-Alg(A).

In the subsequent note it is shown that V-Trip(A) is isomorphic to the dual of V-Alg(A) and that the morphisms of V-Alg(A) are inverse limit preserving V-functors.

0. Introduction

In [5], Frei considers the category of the triples in a category A, and of the triple morphisms, Trip(A), and the category of the categories of algebras $A^{\mathsf{T}}, A^{\mathsf{T}'}, \ldots$ and of the functors which are compatible with the forgetful functors, Alg(A). He shows that Trip(A) is isomorphic to the dual of Alg(A) and that the morphisms of Alg(A) preserve (inverse) limits.

Let V be a symmetric monoidal closed category, in the Eilenberg-Kelly [3] sense. In [3] the V-categories, which are also called enriched categories in Day-Kelly [2], are constructed. The V-triples in V-categories and the V-categories of algebras over V-triples are constructed in Bunge [1]. Then one can consider

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V-Trip(A) , the category whose objects are the V-triples in a V-category A , and V-Alg(A) , the category whose objects are the V-categories of V-algebras over A .

In this note, using the notion of V-limit in the Kelly sense [6] too, we extend in this enriched V-context all the results obtained by Frei.

1. Preliminaries

We use the notions of symmetric monoidal closed category V and V-category, in the Eilenberg-Kelly sense [3]. A V-triple $\mathsf{T} = (T, \eta, \mu)$ in a V-category A consists of a V-functor $T : A \to A$ and two V-natural transformations $\eta : 1_A \to T$, $\mu : TT \to T$ so that:

(i) $\mu.\mu T = \mu.T\mu$, and

(ii) $\mu.T\eta = \mu.\eta T = 1_T$.

A V-triple morphism $\tau : \mathsf{T} \to \mathsf{T}'$ consists of a V-natural transformation $\tau : T \to T'$ so that:

(i) $\tau.\eta = \eta'$, and

(ii) $\tau.\mu = \mu'.\tau T'.T\tau$.

Let A_o be the underlying category of A . A V-triple in A is a triple in A_o too. Thus, one can construct the category of algebras A_o^{T} , whose objects are pairs $[A, \xi]$ where $A \in A$, $\xi \in A_o(TA, A)$, $\xi.\eta_A = 1_A$, $\xi.\mu_A = \xi.T\xi$, and whose morphisms are $[f] : [A, \xi] \to [B, \theta]$ where $f : A \to B$ in A_o and $f.\xi = \theta.Tf$. Bunge ([1], 2.2) shows that A_o^{T} has a V-category structure, if V has equalizers (in what follows we suppose this condition to be fulfilled), given by $A^{\mathsf{T}} : \left(A_o^{\mathsf{T}}\right)^{op} \times A_o^{\mathsf{T}} \to V_o$; in this structure $U^{\mathsf{T}} : A_o^{\mathsf{T}} \to A_o$, given by $[A, \xi] \mapsto A$ and $[f] \mapsto f$, has a V-functor structure and has a V-adjoint F^{T} given at the underlying level by $A \mapsto [TA, \mu_A]$ and $f \mapsto [Tf]$. The functor A^{T} is given on objects by $U^{\mathsf{T}}_{[A,\xi][B,\theta]} : \mathsf{A}^{\mathsf{T}}([A,\ \xi][B,\ \theta]) \to \mathsf{A}(A,\ B)$, the equalizer of the following pair

$$\mathsf{A}(A,\ B) \xrightarrow{T_{AB}} \mathsf{A}(TA,\ TB) \xrightarrow{\mathsf{A}(TA,\ \theta)} \mathsf{A}(TA,\ B)$$
$$\mathsf{A}(A,\ B) \xrightarrow{\mathsf{A}(\xi,\ B)} \mathsf{A}(TA,\ B)$$

In this way the *V-category* A^{T} is well defined.

We recall that $R : \mathcal{B} \to \mathsf{A}$ is a $\mathcal{B}$-adjoint for $S : \mathsf{A} \to \mathcal{B}$ if there exists a V-natural family of isomorphisms $n_{AB} : \mathsf{A}(RB,\ A) \simeq \mathcal{B}(B,\ SA)$ in V_0 , for each $A \in \mathsf{A}$, $B \in \mathcal{B}$.

Let $D : K \to \mathsf{A}_0$ be a functor and let $\mathcal{B}$ be a V-category. The family of morphisms $\left\{L \xrightarrow{\lambda_K} DK\right\}_{K \in \mathsf{K}}$ in $\mathcal{B}_0$ is a *limit in* $\mathcal{B}$ (or a V-limit) in the Kelly sense [6], if the family $\{\mathcal{B}(B,\ L) \to \mathcal{B}(B,\ DK)\}_{K \in \mathsf{K}}$ is a limit in V_0 , for each $B \in \mathcal{B}$.

LEMMA 1.1. *If a V-functor* $T : \mathsf{A} \to \mathcal{B}$ *has a V-adjoint* $S : \mathcal{B} \to \mathsf{A}$ *then* T *preserves the limits of* A .

Proof. Let $\{L \to DK\}$ be a limit in A . Then $\{\mathsf{A}(A,\ L) \to \mathsf{A}(A,\ DK)\}$ is a limit in V_0 , by the above definition, for each $A \in \mathsf{A}$, and so $\{\mathsf{A}(SB,\ L) \to \mathsf{A}(SB,\ DK)\}$ are also limits in V_0 , for each $B \in \mathcal{B}$. Using the V-natural isomorphism n , the families $\{\mathcal{B}(B,\ TL) \to \mathcal{B}(B,\ TDK)\}$ are also limits in V_0 . Hence $\{TL \to TDK\}$ is a limit in $\mathcal{B}$. □

2. The V-isomorphism theorem

PROPOSITION 2.1. *There is a functor* $G : V\text{-Trip}(\mathsf{A}) \to [V\text{-Alg}(\mathsf{A})]^{\mathrm{op}}$ *given by:* $G(\mathsf{T}) = \mathsf{A}^{\mathsf{T}}$ *on objects, and on* $\phi : \mathsf{T} \to \mathsf{T}'$ *by a V-functor* $G\phi : \mathsf{A}^{\mathsf{T}'} \to \mathsf{A}^{\mathsf{T}}$ *, given by* $(G\phi)[A,\ \xi'] = [A,\ \xi'.\phi_A]$ *and by a unique morphism in* V_0

$$(G\phi)_{[A,\xi'][B,\theta']} : \mathsf{A}^{\mathsf{T}'}([A,\ \xi'][B,\ \theta']) \to \mathsf{A}^{\mathsf{T}}([A,\ \xi'.\phi_A][B,\ \theta'.\phi_B])$$

for each $[A, \xi']$, $[B, \theta']$ *in* $\mathsf{A}^{\mathsf{T}'}$.

Proof. Using Diagram 1 on page 379, which commutes, since $U^{\mathsf{T}'}_{[A,\xi'],[B,\theta']}$ is an equalizer and since ϕ is V-natural, one can find

$$\mathsf{A}(\xi'.\phi_A, B)U^{\mathsf{T}'}_{[A,\xi'][B,\theta']} = \mathsf{A}(TA, \theta'.\phi_B).T_{AB}.U^{\mathsf{T}'}_{[A,\xi'][B,\theta']} .$$

$U^{\mathsf{T}}_{[A,\xi'.\phi_A][B,\theta'.\phi_B]}$ being an equalizer it follows that there exists a unique morphism $(G\phi)_{[A,\xi'][B,\theta']}$ so that

$$U^{\mathsf{T}'}_{[A,\xi'][B,\theta']} = U^{\mathsf{T}}_{[A,\xi'.\phi_A][B,\theta'.\phi_B]}.(G\phi)_{[A,\xi'][B,\theta']} .$$

As in [1], the $G\phi$'s defining equality taking place also at the underlying level, it follows that $G\phi$ is a V-functor. Being compatible with the forgetful functors, $G\phi$ is a morphism in $[V\text{-Alg}(\mathsf{A})]^{\mathrm{op}}$. □

PROPOSITION 2.2. *There is a functor* $H : [V\text{-Alg}(\mathsf{A})]^{\mathrm{op}} \to V\text{-Trip}(\mathsf{A})$ *given by* $H(\mathsf{A}^{\mathsf{T}}) = \mathsf{T}$ *and by* $H(R) : \mathsf{T} \to \mathsf{T}'$, *a* V*-natural transformation given in* A_0 *by the family,* $H(R)^{\circ}_A = U_0 a^{-1}_{A,RF'A}(\eta'_A)$ *where* $U^{\mathsf{T}}_0 : \mathsf{A}^{\mathsf{T}}_0 \to \mathsf{A}_0$, $a^{-1}_{A,RF'A} : \mathsf{A}_0(A, T'_0A) \approx \mathsf{A}^{\mathsf{T}}_0(FA, RF'A)$ *are the underlying corresponding notions and we denote* F^{T} *and* $F^{\mathsf{T}'}$ *simply by* F , F' .

Proof. According to [5] it remains to show that $H(R)$ is V-natural. Using the formula $a^{-1}_{X,[Y,\sigma]}(g) = [\sigma.Tg]$ for $g : X \to U^{\mathsf{T}}([Y, \sigma]) = Y$ and by the remark $R([X, \xi']) = [X, R(\xi')]$ one can show that $H(R)^{\circ}_A = R_0(\mu'_A).T_0\eta'_A$. The V-naturality of $H(R)$ follows now from the V-naturality of η' and μ' and from the V-functoriality of R and T ([3], I, 10.2).

THEOREM 2.3. *The functors* G *and* H *are inverse to each other, that is,* $V\text{-Trip}(\mathsf{A})$ *is isomorphic to* $[V\text{-Alg}(\mathsf{A})]^{\mathrm{op}}$.

Proof. As in [5].

$$
\begin{tikzcd}
A^{T'}([A, \xi'], [B, \theta']) \arrow[r, "U^{T'}_{[A,\xi'][B,\theta']}"] & A(A, B) \arrow[r, "T'_{AB}"] \arrow[d, "T_{AB}"] \arrow[rr, bend left, "A(\xi', B)"] & A(T'A, T'B) \arrow[r, "A(T'A, \theta')"] \arrow[d, "A(\varphi_A, T'B)"] & A(T'A, B) \arrow[d, "A(\varphi_A, B)"] \\
 & A(TA, TB) \arrow[r, "A(TA, \varphi_B)"] & A(TA, T'B) \arrow[r, "A(TA, \theta')"] & A(TA, B)
\end{tikzcd}
$$

Diagram 1.

3. The morphisms of V-Alg(A)

PROPOSITION 3.1. *Let* T *be a* V*-triple in* A *and* $U : \mathsf{A}^{\mathsf{T}} \to \mathsf{A}$ *be the corresponding underlying* V*-functor. Then a* V*-functor* $G : \mathsf{B} \to \mathsf{A}^{\mathsf{T}}$ *preserves limits in* B *iff* UG *does.*

Proof. Using Lemma 1.1, the condition is easily seen to be necessary, $F^{\mathsf{T}} : \mathsf{A} \to \mathsf{A}^{\mathsf{T}}$ being a V-adjoint for U . Conversely, let W be an object in V_o , $[A, \xi]$ an object in $\mathsf{A}_o^{\mathsf{T}}$ and $\alpha_K : W \to \mathsf{A}^{\mathsf{T}}([A, \xi], GK)$, $K \in \mathsf{K}$ a compatible (in K) family of morphisms in V_o . We show that there exists a unique morphism $f : W \to \mathsf{A}^{\mathsf{T}}([A, \xi], GL)$ so that $\alpha_K = \mathsf{A}^{\mathsf{T}}([A, \mu], G_o\lambda_K).f$.

In Diagram 2, on page 381, all the squares commute, since $\mathsf{A}(TA, -)$ is a functor, $G_o\lambda_K$ is an algebra morphism and since $\theta : TUGL \to UGL$ and $\theta_K : TUGDK \to UGDK$ are the T-structures of UGL and $UGDK$. Since UG preserves the limits of A , for each $A \in \mathsf{A}$, $\mathsf{A}(A, UGL) \xrightarrow{\mathsf{A}(1,\, U_oG_o\lambda_K)} \mathsf{A}(A, UGDK)$ is a limit in V_o . Since α_K is a compatible family, there exists a unique morphism $g : W \to \mathsf{A}(A, UGL)$ so that $U.\alpha_K = \mathsf{A}(1, U_oG_o\lambda_K).g$ and hence we have another commuting square in Diagram 2.

Now, we show that g equalizes the pair $\mathsf{A}(TA, \theta).T$ and $\mathsf{A}(\xi, UGL)$. Indeed we have

$$\mathsf{A}(1, UG\lambda_K).\mathsf{A}(\xi, UGL).g = \mathsf{A}(\xi, UGDK).\mathsf{A}(1, UG\lambda_K).g = \mathsf{A}(\xi, UGDK).U.\alpha_K =$$
$$\mathsf{A}(TA, \theta_K).T.U.\alpha_K = \mathsf{A}(TA, \theta_K).T.\mathsf{A}(1, UG\lambda_K).g = \mathsf{A}(1, UG\lambda_K).\mathsf{A}(TA, \theta).T.g ,$$

and so the above statement follows from the fact that $\mathsf{A}(1, UG\lambda_K)$ are limits in V_o . Hence there exists a unique morphism $f : W \to \mathsf{A}^{\mathsf{T}}([A, \xi], GL)$ so that $g = U.f$. But,

$$U.\alpha_K = \mathsf{A}(1, UG\lambda_K).g = \mathsf{A}(1, UG\lambda_K).U.f = U\mathsf{A}^{\mathsf{T}}(1, G_o\lambda_K).f ,$$

$$
\begin{tikzcd}
W \arrow[r, "\alpha_K"] \arrow[dr, dashed, "f"'] \arrow[drr, dashed, bend right=40, "g"'] & A^{\mathsf{T}}([A, \xi], GDK) \arrow[r, "U"] & A(A, UGDK) \arrow[r, "T"] \arrow[rr, bend left, "A(\xi, UGDK)"] & A(TA, TUGDK) \arrow[r, "A(TA, \theta_K)"] & A(TA, UGDK) \\
 & A^{\mathsf{T}}([A, \xi], GL) \arrow[u, "A^{\mathsf{T}}(1, G_o\lambda_K)"'] \arrow[r, "U"] & A(A, UGL) \arrow[u, "A(1, U_oG_o\lambda_K)"'] \arrow[r, "T"] \arrow[rr, bend right, "A(\xi, UGL)"'] & A(TA, TUGL) \arrow[u, "A(1, T_oU_oG_o\lambda_K)"'] \arrow[r, "A(TA, \theta)"] & A(TA, UGL) \arrow[u, "A(1, U_oG_o\lambda_K)"']
\end{tikzcd}
$$

Diagram 2.

and so, U being monic, the triangle in Diagram 2 commutes. In this way we already have the existence of the morphism $W \to \mathsf{A}^{\mathsf{T}}([A, \xi], GL)$ we were looking for.

The uniqueness follows from the fact that g is unique for its defining commutation and from the fact that if $f \neq f'$ then $Uf \neq Uf'$ and hence $\mathsf{A}(1, UG\lambda_K).U.f \neq \mathsf{A}(1, UG\lambda_K).U.f'$, the motivation of this last fact being already mentioned above. □

The theorem and corollary which follow are immediate consequences of the last proposition.

THEOREM 3.2. *The morphisms in* V-Alg(A) *preserve limits.*

COROLLARY 3.3. *Let* T *be a* V*-triple in* A *,* $(\gamma, \varepsilon) : S \dashv T : \mathsf{B} \to \mathsf{A}$ *a* V*-adjoint pair which generates* T *, and* $L : \mathsf{B} \to \mathsf{A}^{\mathsf{T}}$ *, the canonical* V*-functor given by* $LB = [TB, T\gamma_B]$ *and by* $L_{BB'}$ *, as in 2.3 in* [1], L *preserves limits.*

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University of Cluj,
Cluj, Roumania.