Very strongly clean rings

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Abstract

An element a in a ring R is *clean* if $a = e + u$ with idempotent e and unit $u \in U(R)$ and strongly clean if $eu = ue$. A clean element is *trivially clean* if the idempotent is *trivial* (i.e. $e \in \{0, 1\}$). A ring is *Abelian* if all idempotents are central.

We start with the following

Definition. An element $a \in R$ is *very strongly clean* (*VSC* for short) if there is a unit $u \in R$ and a *central* idempotent ε such that $a = \varepsilon + u$.

Clearly, *every VSC element is strongly clean* and trivial clean elements are VSC. A ring is *VSC* if all its elements are VSC.

This is not a new class of rings. To clarify this we just need the following

Exercise 1 *Let R be any ring. An idempotent is VSC iff it is central.*

Proof. Suppose an idempotent $e \in R$ is VSC, that is, $e = u+f$ for $u \in U(R)$ and $f^2 = f \in Z(R)$. Then $ef = e(ef) = (u+f)(ef) = uef+ef$ implies $uef = 0$ and so $ef = 0$ (because u is a unit). So $0 = ef = (u+f)f = uf+f$ implies $f = -uf$ and $e = e^2 = (u+f)e = ue+fe = ue$. Hence $ue = e = u+f = u-uf = u(1-f)$ which implies $e = 1 - f$ (u is a unit). So $e \in Z(R)$.

Conversely, if $e \in Z(R)$, $e = (1 - e) + (2e - 1)$ is a well-known VSC decomposition ($(2e - 1)^2 = 1$). ■

Therefore

Theorem 2 *The VSC rings are precisely the Abelian clean rings.*

Proof. One way follows by definitions: indeed, for an Abelian ring, (a) R is VSC; (b) R is strongly clean; (c) R is clean; (d) R is exchange, are equivalent properties. Conversely, one uses the previous exercise (the denial is: a ring with noncentral idempotents is not VSC). ■

Notice that for (unital) rings

$$\text{local} \Longrightarrow \text{VSC} \Longrightarrow \text{strongly} - \text{clean}.$$

None of these implications is reversible, as shown by the following

Examples. **1)** *A VSC ring which is not local.*

Any Boolean ring B is (commutative and) regular, so clean (see Theorem 10, [1]). All elements are idempotent so if $|B| > 2$, it is not local: local rings have only trivial idempotents (see **19.2** [2]).

2) *A strongly clean ring which is not VSC.*

For any field F the ring $\left[\begin{array}{cc} F & F \\ 0 & F \end{array}\right]$ is strongly clean but not VSC.

Indeed, by computation, it has four idempotents: $\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$ which are not central and $0_2, I_2$. So the ring is not VSC by the above exercise.

Moreover, the exercise shows that *matrix rings are not VSC*, so *VSC* (or Abelian clean) *is not a Morita property.*

However, the VSC $n\times n$ matrices are of form $\varepsilon I_n + U$ with central idempotent ε and invertible matrix U.

Remark. As above, an element $a \in R$ is *very strongly nil-clean* (*VSNC* for short) if there is a nilpotent $t \in R$ and a *central* idempotent ε such that $a = \varepsilon + t$.The proof for "strongly nil-clean" implies "strongly clean" can be adapted in order to prove

Exercise 3 *Let R be any ring. An idempotent is VSNC iff it is central.*

Hence

Theorem 4 *The VSNC rings are precisely the Abelian nil-clean rings.*

Professor Lam comments.

Indeed, these facts are well-known. For "Exercise 1", recall that, if $a = e + u$ is any clean decomposition, then we have "Nicholson's equation": $u[u^{-1}(1-e)u - a] = a - a^2$.

If you further assume that $a = e + u$ is a strongly clean decomposition (that is, $ue = eu$), this would simplify to $u[1 - e - a] = a - a^2$. From this, it follows that a is an idempotent iff $a = 1 - e$.

As you said, "Theorem 2" follows immediately from the above. For instance, "Theorem 2" appeared as "Theorem 4.8" in Tuganbaev's survey article "Rings and modules with exchange properties" in the Journal of Mathematical Sciences, Vol. 110 (2002), 2348-2421.

References

[1] D. D. Anderson, V. P. Camillo *Commutative rings whose elements are a sum of a unit and idempotent.* Comm.Algebra, **30** (7) (2002), 3327-3336.

[2] T. Y. Lam *A First Course in Noncommutative Rings.* Graduate Texts in Mathematics 131, Second Edition, Springer 2001.