ON UNIT STABLE RANGE MATRICES

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Abstract. We characterize the unit stable range one 2×2 and 3×3 matrices over commutative rings. In particular, we characterize the 2×2 matrices which satisfy the Goodearl-Menal condition. For 2×2 integral matrices we show that the stable range one and the unit stable range one properties are equivalent, and, that the only matrix which satisfies the Goodearl-Menal condition is the zero matrix.

1. Introduction

The unit stable range one for elements in a unital ring was introduced in [6] and further studied in [3].

Definition. An element a in a ring R is said to have *(left) stable range one* (*sr1*, for short) if, for any b, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb$ is a unit. If r can be chosen to be a unit, we say that a has *(left) unit stable range one* (*usr1*, for short). Right (unit) stable range one is defined symmetrically.

Equivalently, a has *left unit sr1* if for every $x, b \in R$ with $xa + b = 1$, there is a unit $u \in R$, called *unitizer* (as in [1]), such that $a + ub$ is a unit. Equivalently, for every $x \in R$, there is a unit u such that $a + u(1 - xa)$ is a unit.

By left multiplication with $-u^{-1}$ (and change of notation), notice that a has *left unit sr1* if and only if for every $x \in R$, there is a unit u such that

$$(u + x)a - 1$$

is a unit.

So far it is not known whether the stable range one for elements is a left-right symmetric property.

Recall (well-known as the "*Jacobson's Lemma*") that for any unital ring R and elements $\alpha, \beta \in R$, $1 + \alpha\beta$ is a unit if and only if $1 + \beta\alpha$ is a unit. Using the last equivalent definition, it follows that *the unit stable range one (for elements) is a left-right symmetric property.* Therefore, in the sequel we chose to discuss only about *left unit sr1 elements*, removing the "left" attribute. We also use $usr(a) = 1$ to indicate that a has unit sr1. Notice that *zero* has trivially unit sr1.

Also recall the following.

Definition. A ring R is said to satisfy *the Goodearl-Menal condition* (GM, for short; see [6]) provided that for any $x, y \in R$ there exists a unit u such that both $x - u$, $y - u^{-1}$ are units.

In this paper, we specialize this to elements of rings, as follows: an element $a \in R$ *satisfies the GM condition* if for every $x \in R$, there exists a unit u such that both $x - u$, $a - u^{-1}$ are units. Notice that $a - u^{-1}$ is a unit if and only if $ua - 1$ is a unit, which is a special case of the unit sr1 definition, for $x = 0$.

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In Section 2, we give simple properties of elements (and rings) which share these two properties. In section 3, a characterization for unit sr1 2×2 and 3×3 matrices over any commutative ring is given, and we show that unit stable range one and stable range one are equivalent properties for 2×2 integral matrices. Some special cases, including idempotents, nilpotents and matrices with zero second row are also discussed.

The last section is dedicated to the Goodearl-Menal condition. A characterization of the 2×2 matrices over commutative rings which satisfy the GM condition is given, and it is shown that the only integral 2×2 matrix which satisfies the GM condition, is the zero matrix.

All rings we consider are associative and unital. For any unital ring R, $U(R)$ denotes the set of all the units, $N(R)$ denotes the set of all the nilpotents, $J(R)$ denotes the Jacobson radical of R and for any positive integer $n\geq 2$, $\mathbb{M}_n(R)$ denotes the ring of all the $n\times n$ matrices over R. By E_{ij} we denote a square matrix having all entries zero, excepting the (i,j) entry, which is 1. For a matrix A, we denote by $adj(A)$ the adjugate (also called the classical adjoint) matrix.

Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Prerequisites

The following result can be adapted from [6].

Lemma 1. *If a satisfies the GM condition then a has unit sr1.*

Proof. If for every $x\in R$ there is a unit $u\in U(R)$ such that both $x-u$, $a-u^{-1}$ are units, then $(-(x-u)+x)a-1=ua-1=u(a-u^{-1})\in U(R)$, that is, $-(x-u)$ is a unit unitizer for a, as desired. □

Recall that a ring-theoretic property $\mathcal{P}$ is said to be *invariant* to conjugations (or to equivalences), if for any element a having property $\mathcal{P}$ and any units u,v, the element $u^{-1}au$ (resp. uav) also has property $\mathcal{P}$.

Next, a useful result we often use in the sequel.

Lemma 2. *(i) If a has unit sr1 and $v\in U(R)$, then va has unit sr1.*

(ii) If a has unit sr1, so is $-a$.

(iii) Unit sr1 elements are invariant to conjugations.

(iv) If a has unit sr1 and $v\in U(R)$, then av has unit sr1.

(v) Unit sr1 elements are invariant to equivalences.

Proof. (i) Let $x,b\in R$. Suppose $x(va)+b=1$. Then there is u such that $a+ub\in U(R)$. By left multiplication with v, we get $va+vub\in U(R)$, as desired.

(ii) It suffices to take $v=-1$ in (i).

(iii) For every $x\in R$ there is a u such that $a+u(1-xa)\in U(R)$. Then $v^{-1}[a+u(1-xa)]v\in U(R)$ and it follows that $v^{-1}av+v^{-1}uv[1-(v^{-1}xv)(v^{-1}av)]$, as desired.

(iv) If a has unit sr1 and $v\in U(R)$, then $v^{-1}av$ has unit sr1 by (iii). Then by (i), $v(v^{-1}av)=av$ has unit sr1.

(v) Follows from (i) and (iv). □

Further, recall from [12] that an element a in a ring R is called *2-good* provided that it is a sum of two units. A ring R is *2-good* if so are all its elements.

Any unit sr1 element partly satisfies the GM condition.

Lemma 3. *(i) If $usr(a) = 1$, there exists a unit u such that $a - u^{-1} \in U(R)$.*
(ii) If $usr(a) = 1$, then a is 2-good.

Proof. (i) Since for every $x \in R$ there is a unit $u \in U(R)$ such that $(u + x)a - 1 \in U(R)$, we just take $x = 0$. Hence $ua - 1 \in U(R)$ and so $a - u^{-1} = u^{-1}(ua - 1) \in U(R)$.

(ii) This follows from (i), since $a - u^{-1} \in U(R)$, for a unit u iff a is 2-good. □

Therefore we have the following implications

$$a \text{ satisfies } GM \Rightarrow usr(a) = 1 \Rightarrow a \text{ is 2-good.}$$

Examples in $\mathbb{M}_2(\mathbb{Z})$ show that these implications are irreversible: having zero determinant, E_{11} has usr1 (see Proposition 5) but does not satisfy the GM condition (see last section, final step of the proof of Theorem 17), and $2I_2 = I_2 + I_2$ is clearly 2-good, but does not have even sr1 (see [1]: the matrix nI_2 has sr1 iff $n \in \{-1, 0, 1\}$).

Taking $r = 0$ in the first definition of Introduction shows that *units have sr1.* However, *units may not have unit sr1.* Easy examples are ± 1 in $\mathbb{Z}$. However, we have the following characterization.

Lemma 4. *In a ring R, $usr(1) = 1$ iff R is 2-good.*

Proof. According to the equivalent definition, $usr(1) = 1$ iff for every $x \in R$, there is a unit u such that $u + x - 1 \in U(R)$. Hence $x - 1 \in U(R) + U(R)$ and so $R = U(R) + U(R)$. Conversely, for every $x \in R$ there are units $u, v \in U(R)$ such that $x - 1 = u + v$. Hence $-u + x - 1 = v \in U(R)$ and so $usr(1) = 1$. □

Since $\mathbb{Z}$ is not 2-good, using also Lemma 2 (ii), we infer that 0 is the only unit sr1 element of $\mathbb{Z}$. Hence, $\mathbb{Z}$ is an example of ring whose *units do not have unit sr1.*

3. Unit stable range 1 matrices

We start with an example of unit sr1 (square) matrix over any ring.

Proposition 5. *Let R be any ring. For any positive integer n and any $r \in R$, the matrices rE_{ij} have unit sr1 in $\mathbb{M}_n(R)$.*

Proof. In [1, Proposition 8,(ii)], using the invariance to equivalences, it was proved that the matrices rE_{ij} have sr1. More precisely, using two permutation matrices, it was sufficient to show that rE_{11} has sr1. For rE_{11} and for any $X \in \mathbb{M}_n(R)$, a suitable unitizer (see definition in the Introduction) is

$$Y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & (-1)^n a_1 \end{bmatrix},$$

where $\mathrm{col}_1(X) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Since by Lemma 2, unit sr1 is also invariant to equivalences and this unitizer is a unit, the statement follows. □

Unitizers are not unique. Out of the unitizers provided by the previous proposition, here are some other.

Examples. 1) *Over any ring, rE_{12} has usr1.*

For every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can also take the unitizer $U = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$. Then $(U + X)rE_{12} - I_2 = \begin{bmatrix} -1 & (a + 1)r \\ 0 & -1 \end{bmatrix}$ is a unit and so rE_{12} has unit sr1.

2) *Over any ring, rE_{11} has usr1.*

For every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can also take the unitizer $U = \begin{bmatrix} -a & 1 \\ -1 & 0 \end{bmatrix}$. Then $(U + X)rE_{11} - I_2 = \begin{bmatrix} -1 & 0 \\ r(c - 1) & -1 \end{bmatrix}$ is a unit and so rE_{11} has unit sr1.

Recall that a ring is called *GCD* (see [5]) if every pair of elements has a gcd (greatest common divisor).

Corollary 6. *Over any GCD ring R, nilpotents and idempotents $\neq I_2$ have unit sr1 in $\mathbb{M}_2(R)$.*

Proof. This follows, since over any GCD ring, every 2×2 nilpotent matrix is similar to rE_{12} for some $r \in R$, and every nontrivial idempotent matrix is similar to E_{11} (see for example [2], Corollary 1 and Corollary 2). $\square$

In trying to generalize this result for idempotents, recall that following Steger [11], we say that a ring R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Thus, by a result of Song and Guo [10], if every idempotent matrix over R is equivalent to a diagonal matrix, then R is an ID ring. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings (see [8]), unit-regular rings and serial rings.

Therefore, due to the invariance to equivalences, over an ID ring, the determination of the idempotent matrices which have unit sr1, reduces to diagonal matrices.

Corollary 7. *Over any elementary divisor ring R, idempotents of $\mathbb{M}_n(R)$ have unit sr1.*

Proof. Using the Song and Guo result mentioned above and the invariance of unit sr1 to equivalences, it suffices to check all block matrices $E = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & 0_{n-m} \end{bmatrix}$. The statement reduces to the fact that E_{11} has unit sr1 in $\mathbb{M}_n(R)$ for any $n \geq 2$. Indeed, $E = E_{11} + ... + E_{mm} = \begin{bmatrix} I_{m-1} & 0 \\ 0 & E_{11} \end{bmatrix}$, where $E_{11} \in \mathbb{M}_{n-m+1}(R)$ has unit sr1 by Proposition 5 and $I_{m-1} \in \mathbb{M}_{m-1}(R)$ has unit sr1 by Lemma 4 and the well-known result (see [12], Proposition 6): any proper matrix ring over an elementary divisor ring is 2-good. So $E = \begin{bmatrix} I_{m-1} & 0 \\ 0 & E_{11} \end{bmatrix}$ has unit sr1 in $\mathbb{M}_n(R)$. $\square$

Remark. Note that for any ring R and for any positive integers m, n, if both $\mathbb{M}_m(R)$ and $\mathbb{M}_n(R)$ have unit 1-stable range, then so does $\mathbb{M}_{m+n}(R)$ (see [3, Theorem 2.2.4]).

As our first main result, we give a characterization of unit stable range one for 2×2 and 3×3 matrices over any commutative ring.

Theorem 8. *(i) Let R be a commutative ring and $A \in \mathbb{M}_2(R)$. Then A has unit stable range 1 iff for any $X \in \mathbb{M}_2(R)$ there exists a unit $U \in \mathbb{M}_2(R)$ such that*

$$\det((U+X)A) - \mathrm{Tr}((U+X)A) + 1$$

is a unit of R.

(ii) Let R be a commutative ring and $A \in \mathbb{M}_3(R)$. Then A has unit stable range 1 iff for any $X \in \mathbb{M}_3(R)$ there exists a unit $U \in \mathbb{M}_3(R)$ such that

$$\det((U+X)A) - \mathrm{Tr}(adj(U+X)A) + \mathrm{Tr}((U+X)A) - 1$$

is a unit of R, where, for any matrix $B \in \mathbb{M}_3(R)$, $adj(B)$ denotes the adjugate of B.

Proof. (i) Using the equivalent definition given in the Introduction, A has unit stable range 1 iff for any $X \in \mathbb{M}_2(R)$ there is a unit $U \in \mathbb{M}_2(R)$ such that $(U+X)A-I_2$ is a unit. Since the base ring is supposed to be commutative, $(U+X)A-I_2$ is invertible iff $\det((U+X)A-I_2))$ is a unit of R. Since for any 2×2 matrix C, $\det(C-I_2) = \det(C) - \mathrm{Tr}(C) + 1$, the statement follows.

(ii) The proof is analogous, relying on the formula $\det(C - I_3) = \det(C) - \mathrm{Tr}(adj(C)) + \mathrm{Tr}(C) - 1$, where C is any 3×3 matrix. □

As this was done in [1, Corollary 6] for sr1 and 2×2 matrices, we obtain alternative proofs for the left-right symmetry of the unit sr1.

Corollary 9. *Let R be a commutative ring and $A \in \mathbb{M}_2(R)$ or $A \in \mathbb{M}_3(R)$. Then A has left unit stable range 1 iff A has right unit stable range 1.*

Proof. Using the properties of determinants, the properties of the trace and the commutativity of the base ring, it is readily seen that for 2×2 matrices, $\det(U+X)A - \mathrm{Tr}(U+X)A + 1 = \det A(U+X) - \mathrm{Tr} A(U+X) + 1$. For 3×3 matrices C, D we just recall the known formulas

$$adj(C) = \frac{1}{2}(\mathrm{Tr}^2(C) - \mathrm{Tr}(C^2)) - C\mathrm{Tr}(C) + C^2$$

and $\mathrm{Tr}((CD)^2) = \mathrm{Tr}((DC)^2)$.

Hence $\mathrm{Tr}(adj(CD)) = \frac{1}{2}(\mathrm{Tr}^2(CD) - \mathrm{Tr}((CD)^2)) = \frac{1}{2}(\mathrm{Tr}^2(DC) - \mathrm{Tr}((DC)^2) = \mathrm{Tr}(adj(DC))$. Finally, this shows that $\det((U+X)A - I_2)) = \det(A(U+X) - I_2))$, as desired. □

In the sequel, we use the notation $\mathrm{diag}(r,s) := \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$. Next, another useful result we often use in the sequel.

Lemma 10. *Over any ring R, the following statements hold.*

(i) An $n \times n$ matrix A has unit sr1 iff its transpose A^T has it.

(ii) $\mathrm{diag}(r,s)$ has unit sr1 iff $\mathrm{diag}(s,r)$ has it.

(iii) $\mathrm{diag}(r,s)$ has unit sr1 iff $\mathrm{diag}(r,-s)$ has it.

Proof. (i) Indeed, using the left-right symmetry of the unit sr1 property (i.e., Corollary 9), if $(U+X)A - I_2$ is a unit, so is its transpose $A^T(U^T + X^T) - I_2$.

(ii) Follows from Lemma 2 (iii), by conjugation with the involution $E_{12} + E_{21}$.

(iii) Follows from Lemma 2 (v), since $\mathrm{diag}(r,-s)$ is equivalent to $\mathrm{diag}(r,s)$. □

Recall that a commutative unital ring R is an *elementary divisor* ring provided every matrix over R is equivalent to a diagonal matrix. Elementary divisor rings include PIDs, left PIDs which are Bézout (in particular division rings), valuation rings and the ring of entire functions.

The definition above, given by M. Henriksen (see [7]), is more general than the one given by I. Kaplansky in [8] and is nowadays in use.

We have already mentioned that *any proper matrix ring over an elementary divisor ring is 2-good*. Over an Euclidean domain, proper $n \times n$ matrix rings are even *strongly 2-good* (i.e., every matrix is a sum of two invertible matrices that are generated - in $GL_n(R)$ - by elementary matrices or permutation matrices or $-I_n$). Hence, in particular, $\mathbb{M}_2(\mathbb{Z})$ is (strongly) 2-good.

Our second main result is a bit surprising.

Theorem 11. *Let A be an integral 2×2 matrix. The following conditions are equivalent*

(i) A has unit sr1,

(ii) A has sr1,

(iii) $\det(A) \in \{-1, 0, 1\}$.

Proof. (i) $\Rightarrow$ (ii) is obvious and (ii) $\Leftrightarrow$ (iii) was proved in [1] (see Theorem 11). Since both properties in (iii) and (i) are invariant to equivalences (over $\mathbb{Z}$), and $\mathbb{Z}$ is an elementary divisor ring, for (iii) $\Rightarrow$ (i), it only remains to check that diagonal 2×2 matrices with $\det(A) \in \{-1, 0, 1\}$ have unit sr1. But these are 0_2, $\pm I_2$, $\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and matrices nE_{11} or mE_{22} with nonzero integers n, m. Since 0 has unit sr1 in any ring, using Lemma 2, (ii) and Lemma 10, and using Proposition 5 for matrices with three zero entries, it only remains to show that I_2 has unit sr1. But this follows from Lemma 4, since $\mathbb{M}_2(\mathbb{Z})$ is (even strongly) 2-good. $\square$

Recall that for a ring-theoretic property $\mathcal{P}$, an element a in a ring R has the *complementary property* if $1 - a$ has $\mathcal{P}$ whenever a has it.

Since the following properties were proved in [1] for sr1 matrices (see Propositions 14 and 15), we deduce these at once also for unit sr1.

Corollary 12. *(i) In general, unit stable range 1 elements do not have the complementary property.*

(ii) In $\mathbb{M}_2(\mathbb{Z})$, AB has unit stable range 1 iff BA has it.

(iii) Jacobson's Lemma holds for unit stable range 1 matrices in $\mathbb{M}_2(\mathbb{Z})$.

Remarks. 1) We infer from the above theorem that if an integral 2×2 matrix has a unitizer (i.e., has sr1), then it has also a unitizer which is a unit (i.e., has unit sr1).

2) In view of [1], the previous proof reduces to check that I_2 has unit sr1. Here is **a direct proof** for this.

According to the equivalent definition given in the Introduction, for every integral 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we should indicate a unit unitizer $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $U + X - I_2 = V$ is a unit of $\mathbb{M}_2(\mathbb{Z})$. As already mentioned in Lemma 4, this is done in two steps.

Step 1. We diagonalize $X - I_2$ by elementary operations, that is, find d_1, d_2 such that $X - I_2$ is equivalent to $\mathrm{diag}(d_1, d_2)$. Over $\mathbb{Z}$ this can be done, using the Euclid's algorithm (see e.g., [8] or [13]). We just sketch this for reader's convenience.

Using elementary row and column operations, we can equivalently replace $X - I_2$ by a matrix M, which we also denote by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that a is the least entry in absolute value, among all matrices that $X - I_2$ can be reduced to. Next, if $b = aq_1 + r_1$, $a = r_1q_2 + r_2$ it can be shown that $r_2 = 0$ (by the minimality in absolute value of a). Hence we can assume $b = aq + r$, $a = rs$ for some integers q, r, s. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} r & 0 \\ * & * \end{bmatrix}$, reduces our matrix to (lower) triangular form. In a similar way, this matrix is reduced to a matrix of form $\begin{bmatrix} * & * \\ 1 & 0 \end{bmatrix}$, which is easily diagonalized.

Step 2. Since $\mathrm{diag}(d_1, d_2) = \begin{bmatrix} d_1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & d_2 \end{bmatrix} = -U + V$ is 2-good, these units suit for our above purpose.

Clearly, all this can be done for 2×2 matrices over any Euclidean domain.

Therefore, it would be difficult to hope for a formula which gives in general the unit unitizer U, given X.

Examples. 1) Matrices $M_{uv} = \begin{bmatrix} 1 & u \\ v & uv \end{bmatrix}$ have unit sr1 over any commutative ring.

Indeed, M_{uv} having zero determinant, according to Theorem 8 (i), it suffices to find a unit unitizer $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ for which $1 - Tr[(U + X)M_{uv}]$ is a unit. For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, an easy choice is $x = -a + u - v - cu - vb - uvd$, $y = 1$, $z = -1$, $t = 0$, which gives $Tr[(U + X)M_{uv}] = 0$ and the unit unitizer $U = \begin{bmatrix} -a + u - v - cu - vb - uvd & 1 \\ -1 & 0 \end{bmatrix}$.

2) Simple examples show that $\mathbb{M}_2(\mathbb{Z})$ is *not* closed under addition of unit sr1 matrices. Indeed, E_{11}, I_2 both have unit sr1, but the (diagonal) sum has not.

Our third main result is the following.

Theorem 13. *Let R be a commutative ring. All matrices in $\mathbb{M}_2(R)$ with (at least) one zero row or zero column have unit sr1 in any of the following cases*

(i) one entry divides the other;

(ii) R is an Euclidean domain;

(iii) for every $a, c \in R$ there are $q, \alpha, \beta \in R$ such that $(a + qs)\alpha + (-c + qr)\beta = 1$ (e.g., R is a Bézout domain).

Proof. By Lemma 10 (i), it suffices to prove the claim for matrices with zero second row.

(i) Let $A = \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$ with $r, s \in R$ and suppose s divides r. Since $\det(A) = 0$, replacement in Theorem 8, (i) gives $-\mathrm{Tr}((U + X)A) + 1 \in U(R)$ or $1 - r(a + x) - s(c + z) \in U(R)$. If $1 - r(a + x) - s(c + z) = 1$, then we have to solve

$rx + sz = -ra - sc$ with $xt - yz = 1$. We can eliminate z, multiplying $xy - yz = 1$ by s. We obtain $sxt + y(rx + ra + sc) = s$, which has solution $x = t = 1$, $y = 0$ and $z = -c - \frac{r}{s}(a+1)$. Hence $U = \begin{bmatrix} 1 & 0 \\ -c - \frac{r}{s}(a+1) & 1 \end{bmatrix}$ is a suitable unit unitizer.

If r divides s, similarly a suitable unit unitizer is $U = \begin{bmatrix} -a + \frac{s}{r}(1-c) & 1 \\ -1 & 0 \end{bmatrix}$.

(ii) Write $r = sq_1 + r_1$, $s = r_1q_2 + r_2$, ..., $r_{n-2} = r_{n-1}q_n + r_n$, $r_{n-1} = r_nq_{n+1} + 0$, for the Euclidean algorithm where $r_n = \gcd(r; s)$. Next, notice that $\begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -q_1 \end{bmatrix} = \begin{bmatrix} s & r_1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} s & r_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -q_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 0 & 0 \end{bmatrix}$, ..., $\begin{bmatrix} r_{n-2} & r_{n-1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & r_n \\ 0 & 0 \end{bmatrix}$, whence $\begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$ is equivalent to $\begin{bmatrix} r_{n-1} & r_n \\ 0 & 0 \end{bmatrix}$. Since r_n divides r_{n-1}, (i) applies.

(iii) For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assume $(a+qs)\alpha + (-c+qr)\beta = 1$, for some $q, \alpha, \beta \in R$ and take the (unit) unitizer $U = \begin{bmatrix} -a - qs & \beta \\ -c + qr & -\alpha \end{bmatrix}$. Then $(U+X)\begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix} - I_2 = \begin{bmatrix} -qrs - 1 & -qs^2 \\ qr^2 & qrs - 1 \end{bmatrix}$ is a unit which has determinant one. □

4. The Goodearl Menal condition

Recall that $a \in R$ satisfies the GM condition (rings with the GM condition were also called "*rings with many units*", for obvious reasons) if for every $x \in R$, there is $u \in U(R)$ such that both $x - u, a - u^{-1} \in U(R)$.

As already mentioned, in any ring R, $a - u^{-1} \in U(R)$ is equivalent to $ua - 1 \in U(R)$.

Remark. For every $x \in R$, there is unit u such that $x - u \in U(R)$, is equivalent to R being 2-good. If in addition, $U(R) \cdot N(R) \subseteq N(R)$ (e.g., reduced or commutative rings), then all nilpotents satisfy GM.

The following simple result will be useful.

Lemma 14. *(i) The GM condition is invariant to equivalences.*
(ii) If a unit $a \in U(R)$ satisfies GM then 1 satisfies GM.

Proof. (i) Let $v \in U(R)$ and suppose $a \in R$ satisfies GM. We show that va also satisfies GM. For an arbitrary $y \in R$ consider $x = yv$. By hypothesis, there exists $u \in U(R)$ such that $x - u, a - u^{-1} \in U(R)$ and so $yv - u \in U(R)$ and $a - u^{-1} \in U(R)$. By right multiplication with v^{-1} and left multiplication with v, respectively, we obtain $y - uv^{-1} \in U(R)$ and $va - vu^{-1} = va - (uv^{-1})^{-1} \in U(R)$, as desired.

A symmetric proof shows that also av satisfies GM.

(ii) Follows from (i) by left (or right) multiplication with a^{-1}. □

Remarks. (i) Notice that a satisfies GM iff $-a$ satisfies GM (by multiplication with the unit -1).

(ii) The GM condition is also invariant to conjugations.

Over any commutative ring, we first prove a characterization of the 2×2 matrices which have the GM condition.

Theorem 15. *A 2×2 matrix A over a commutative ring satisfies the GM condition iff for every X there is a unit U such that* $\det(U)$, $\det(X)+\det(U)-\mathrm{Tr}(adj(X)U)$ *and* $\det(U)\det(A)-\mathrm{Tr}(UA)+1$ *are units of R.*

Proof. Analogous to the proof of Theorem 8. □

For the special case of integral matrices we have the following result.

Corollary 16. *An integral matrix A satisfies the GM condition iff for every X there is a unit U such that*

(a) $\mathrm{Tr}(UA)\in\{-1,1,3\}$, $\det(X)-\mathrm{Tr}(adj(X)U)\in\{-2,0,2\}$ *and A is a unit, or else*

(b) $\mathrm{Tr}(UA)\in\{0,2\}$, $\det(X)-\mathrm{Tr}(adj(X)U)\in\{-2,0,2\}$ *and* $\det(A)=0$.

Proof. For $X=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$, $U=\begin{bmatrix} x & y \\ z & t\end{bmatrix}$ and any 2×2 matrix A, both $X-U$ and $UA-I_2$ are units, iff

$$\begin{aligned} \det(U) &\in \{\pm 1\},\\ \det(X)+\det(U)-\mathrm{Tr}(adj(X)U) &\in \{\pm 1\},\\ \det(U)\det(A)-\mathrm{Tr}(UA)+1 &\in \{\pm 1\}.\end{aligned}$$

Case 1. Let $\det(U)=1$. Then $\det(X)-\mathrm{Tr}(adj(X)U)\in\{-2,0\}$ and $\det(A)-\mathrm{Tr}(UA)+1\in\{\pm 1\}$. The first condition is independent from A.

(a) Since A is supposed to be a unit, $1-\mathrm{Tr}(UA)\pm 1\in\{\pm 1\}$, that is, $\mathrm{Tr}(UA)\in\{-1,1,3\}$.

(b) If $\det(A)=0$, we get $\mathrm{Tr}(UA)\in\{0,2\}$.

Case 2. Let $\det(U)=-1$. Then $\det(X)-\mathrm{Tr}(adj(X)U)\in\{0,2\}$ and $-\det(A)-\mathrm{Tr}(UA)+1\in\{\pm 1\}$.

(a) Since A is supposed to be a unit, $1-\mathrm{Tr}(UA)\mp 1\in\{\pm 1\}$, that is, the same $\mathrm{Tr}(UA)\in\{-1,1,3\}$.

(b) If $\det(A)=0$, again we get $\mathrm{Tr}(UA)\in\{0,2\}$. □

It is easy to see that 0 satisfies the GM condition in a ring R iff R is 2-good. Therefore, 0_2 satisfies GM in $\mathbb{M}_2(\mathbb{Z})$.

Our fourth and last main result shows that 0_2 is the only matrix of $\mathbb{M}_2(\mathbb{Z})$ which satisfies GM.

Theorem 17. *Nonzero matrices of $\mathbb{M}_2(\mathbb{Z})$ do not satisfy GM.*

Proof. As noticed in Section 2, elements which satisfy the GM condition, have (unit) sr1. Hence, with respect to integral 2×2 matrices (see also Theorem 11), these are units or have zero determinant. Therefore we split the proof in two parts.

We first show that *units do not satisfy GM in $\mathbb{M}_2(\mathbb{Z})$*. Using Lemma 14, it suffices to show that I_2 does not satisfy GM in $\mathbb{M}_2(\mathbb{Z})$.

According to Corollary 16, I_2 has GM iff for every X there is a unit U such that $\mathrm{Tr}(U)\in\{-1,1,3\}$ and $\det(X)-\mathrm{Tr}(adj(X)U)\in\{-2,0,2\}$.

Here $adj(X)=\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$, so the second condition becomes $ad-bc-dx+bz+cy-at\in\{-2,0,2\}$.

Hence $x+t \in \{-1,1,3\}$, $xt-yz=1$ and $dx-cy-bz+at = ad-bc+k$ with $k \in \{-2,0,2\}$, which are Diophantine linear equations (for which the solvability condition is well-known).

Case 1. $x+t=-1$ or $t=-x-1$ gives

$$(d-a)x-cy-bz = a(d+1)-bc+k.$$

If we take $X = \begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix}$, then $\gcd(d-a;c;b)=5$ and $a(d+1)-bc = 16-25=-9$. Since $-9+k$ is not divisible by 5, there are no integer solutions.

Case 2. $x+t=1$ or $t=1-x$ gives

$$(d-a)x-cy-bz = a(d-1)-bc+k.$$

If we take $X = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}$, then $\gcd(d-a;c;b)=5$ and $a(d-1)-bc = 21-25=-4$. Since $-4+k$ is not divisible by 5, there are no integer solutions.

Case 3. $x+t=3$ or $t=3-x$ gives

$$(d-a)x-cy-bz = a(d-3)-bc+k.$$

If we take $X = \begin{bmatrix} 4 & 5 \\ 5 & 9 \end{bmatrix}$, then $\gcd(d-a;c;b)=5$ and $a(d-3)-bc = 24-25=-1$. Since $-1+k$ is not divisible by 5, there are no integer solutions.

Secondly, we show that the only *zero determinant matrix* which satisfies GM in $\mathbb{M}_2(\mathbb{Z})$ is the zero matrix.

Since $\mathbb{Z}$ is an elementary divisor ring, every matrix over $\mathbb{Z}$ is equivalent to a diagonal matrix, and using Lemma 14, it suffices to prove our claim for diagonal matrices of zero determinant. Excepting the zero matrix and (if necessary) using Lemma 10 (i), it suffices to check this for nE_{11}. Since nE_{11} satisfies GM iff $-nE_{11}$ satisfies GM, we can assume n a positive integer.

For multiples $A = nE_{11}$ with $n \geq 2$, take $X = \begin{bmatrix} 0 & 4 \\ 4 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $\mathrm{Tr}(UA) = nx \in \{0,2\}$ holds only for $x=0$, if $n \geq 3$ and for $x \in \{0,1\}$, if $n=2$. Further, $\det(X) - \mathrm{Tr}(adj(X)U) = -16-3x+4y+4z$. Then if $x=0$, since $xt-yz = \det(U) \in \{\pm 1\}$, $yz \in \{\pm 1\}$ follows and so $y,z \in \{\pm 1\}$. The condition becomes $4(-4+y+z) \in \{-2,0,2\}$, impossible for $y,z \in \{\pm 1\}$. If $x=1$, then $-19+4(y+z-t) \notin \{-2,0,2\}$.

Finally, we show that $A = E_{11}$, does not satisfy GM. With an arbitrary $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, the conditions $\det(X) - \mathrm{Tr}(adj(X)U) \in \{-2,0,2\}$ and $\mathrm{Tr}(UA) \in \{0,2\}$ become $x \in \{0,2\}$ and $ad-bc-dx+cy+bz-at \in \{-2,0,2\}$.

(i) For $x=0$, we have $ad-bc+cy+bz-at \in \{-2,0,2\}$, which are Diophantine equations

$$cy+bz-at = -ad+bc+k$$

where $k \in \{-2,0,2\}$.

If $\delta = \gcd(c;b;a)$ and $\delta \geq 3$, the equations are solvable only if $k=0$. In this case, δ divides also $-ad+bc$, so this equation has always solutions. However, we need a solution such that U is a unit, i.e., $xt-yz \in \{\pm 1\}$.

Take $X = \begin{bmatrix} 15 & 3 \\ 3 & 0 \end{bmatrix}$ and divide a, b, c by $\delta = 3$. Then we obtain the Diophantine equation $y + z - 5t = 1$. Since, as already mentioned $y, z \in \{\pm 1\}$, we have $y + z \in \{-2, 0, 2\}$, so the equation has no solutions.

If $\delta = 2$, the equations are solvable for any $k \in \{-2, 0, 2\}$. Dividing by 2, we can assume that a, b, c are coprime and $k \in \{-1, 0, 1\}$. Again $y, z \in \{\pm 1\}$, whence $2y + 5z \in \{-7, -3, 3, 7\}$. Starting with $X = \begin{bmatrix} 22 & 10 \\ 4 & 0 \end{bmatrix}$, we get $2y + 5z - 11t = -11d + 10 + k$, with $k \in \{-1, 0, 1\}$. The LHS is congruent (mod 11) to 3 or 4 or 7 or 8, but the RHS is congruent to 0 or 9 or 10. So the equation has no solutions.

If $\delta = 1$, the equations are solvable for any $k \in \{-2, 0, 2\}$. Again $y, z \in \{\pm 1\}$, whence $y+2z$ is odd. Starting with $X = \begin{bmatrix} 8 & 2 \\ 1 & 0 \end{bmatrix}$, we get $y+2z-8t = -8d+2+k$. Since LHS is odd and RHS is even, the equation has no solutions.

(ii) For $x = 2$ we have

$$cy + bz - at = (2 - a)d + bc + k$$

with $k \in \{-2, 0, 2\}$. We take $X = \begin{bmatrix} 5 & 5 \\ 5 & 2 \end{bmatrix}$ and so $\gcd(c; b; a) = 5$, but $(2 - a)d + bc = -6 + 25 = 19$ and $19 + k$ is not divisible by 5. This completes the proof. □

5. Open questions

1) Give examples of an element of $J(R) \setminus N(R)$ which has not unit sr1.

Hint: *nilpotents in the Jacobson radical have unit sr1.* Indeed, since $a \in J(R)$ iff $1 - xay \in U(R)$ for any $x, y \in R$ (see [9], Lemma 4.3), for the (unit) sr1 property, we can choose the (unit) unitizer $y = -(1 - a)(1 - xa)^{-1}$, where $1 - a \in U(R)$ if $a \in N(R)$.

2) Find two elements which have unit sr1 but whose product has not unit sr1.

Hint: according to Theorem 11, and the proof for the multiplicative closure of the set of all the sr1 elements (see [4], Lemma 17), such an example cannot be given in $\mathbb{M}_2(\mathbb{Z})$.

3) Which idempotents have unit sr1 ? If R is 2-good, do all idempotents have unit sr1 ?

Idempotents are unit-regular and unit-regular elements have sr1. As already noticed, the idempotent 1 has not unit sr1 in $\mathbb{Z}$.

For matrix rings these questions were addressed in Section 3.

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I declare that there is no conflict of interest in this study.

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