UNIT-REGULAR ELEMENTS AND JACOBSON RADICAL

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ABSTRACT. We show that zero is the only unit-regular element which belongs to the Jacobson radical. We also determine the rings all whose unit-regular elements are idempotents or units.

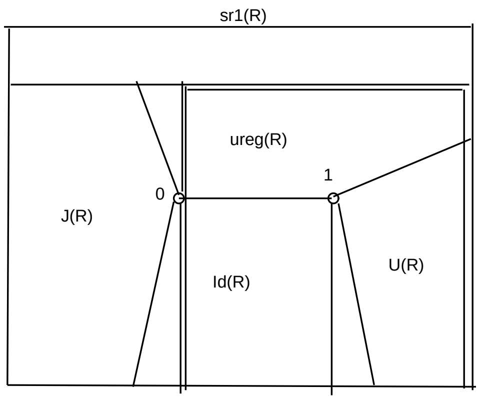
All rings we consider are associative and unital (i.e., with identity). As customarily we denote by $J(R)$ the Jacobson radical, by $U(R)$ the group of units, by $Id(R)$ the set of idempotents and by $ureg(R)$ the set of unit-regular elements of a ring R. By $Id^*(R)$ we denote the set of nonzero idempotents. Moreover, we denote by $sr1(R)$ the (left) stable range one elements of a ring R. The following inclusions are well-known: $Id(R), U(R) \subseteq ureg(R) \subseteq sr1(R)$ and $J(R) \subseteq sr1(R)$. Moreover, since $J(R)$ is an ideal $\neq R$, $U(R) \cap J(R) = \varnothing$ and since the Jacobson radical contains no nonzero idempotents $Id(R) \cap J(R) = \{0\}$.

It is easy to see that

Proposition 1. *The only unit-regular element which belongs to the Jacobson radical is zero.*

Proof. Indeed, let $a \in ureg(R) \cap J(R)$. If $a = aua$ with $u \in U(R)$ then $au \in J(R)$. As au is idempotent we get $au = 0$ and so $a = 0$. □

Therefore the situation is described below



Theorem 2. *For a ring R, $ureg(R) = Id(R) \cup U(R)$ iff R is connected or else $U(R) = \{1\}$.*

Proof. Since $ureg(R) = \{0\} \cup Id^*(R)U(R)$, the ring has the property iff $\{0\} \cup Id^*(R)U(R) = Id(R) \cup U(R)$, and since $Id(R), U(R) \subseteq Id(R)U(R)$ and $Id(R) \cap U(R) = \{1\}$, iff $Id^*(R)U(R) \subseteq U(R)$ or $Id(R)U(R) \subseteq Id(R)$. Equivalently, iff every product eu with $e^2 = e$ and $u \in U(R)$ is either a unit or an idempotent.

In the first situation, since $Id^*(R) \subseteq Id^*(R)U(R) \subseteq U(R)$, it follows that $Id(R) = \{0, 1\}$, that is, R is connected.

In the second situation, since $U(R) \subseteq Id(R)U(R) \subseteq Id(R)$ it follows that $U(R) = \{1\}$.

The converses are straightforward: $\{0\} \cup U(R) = ureg(R)$ in the first case and $Id(R) = ureg(R)$ in the second case. □

Remark. The rings with $U(R) = \{1\}$ are reduced and $char(R) = 2$.

Example. $R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ has $J(R) = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} = N(R)$.

$Id^*(R) = \{I_2\} \cup \left\{ \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Q} \right\}$ and $U(R) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, c \in \{\pm 1\}, b \in \mathbb{Q} \right\}$ (where $\begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}^{-1} = \begin{bmatrix} \pm 1 & \mp b \\ 0 & \pm 1 \end{bmatrix}$).

Finally, for $ureg(R)$ we compute $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ which amounts to $a^2 x = a$, $c^2 z = c$ and $b(ax + cz) + acy = b$ together with $xz = \pm 1$.

If $a \neq 0 \neq c$ then $x = \frac{1}{a}$, $z = \frac{1}{c}$ so $ac \in \{\pm 1\}$ and arbitrary b, so we recover the units.

If both $a = c = 0$ we recover the Jacobson radical. Finally, if (say) $a \neq 0$ and $c = 0$, then $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & y \\ 0 & a \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ so $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in ureg(R)$ for every b. For instance $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ belong to $ureg(R)$ but not to $Id^*(R) \cup U(R)$.