Unipotent 2×2 matrices over commutative rings.

Grigore Călugăreanu

2017

A unit in a ring with identity is called *unipotent* if it has the form $1 + t$ with nilpotent t.

In what follows we consider 2×2 matrices over a commutative ring R.

Lemma 1 *A* 2×2 *matrix* U *is unipotent iff* $\det(U) = 1$ *and* $\mathrm{Tr}(U) = 2$.

Proof. By Cayley-Hamilton theorem, an invertible matrix U is unipotent iff $U - I_2$ has zero trace and zero determinant (i.e. is nilpotent in $\mathcal{M}_2(R)$).

If we denote $U = (u_{ij})$ then for $U - I_2 = \begin{bmatrix} u_{11} - 1 & u_{12} \\ u_{21} & u_{22} - 1 \end{bmatrix}$ we have $\mathrm{Tr}(U - I_2) = \mathrm{Tr}(U) - 2$ and $\det(U - I_2) = \det(U) - \mathrm{Tr}(U) + 1$. The conditions above give $\mathrm{Tr}(U) = 2$ and $\det(U) = 1$. ■

Remark. 1) It is clear that powers of unipotents are also unipotents. In our special case this can also be seen by using the previous lemma.

If $\det(U) = 1$ then clearly $\det(U^n) = 1$. As for the traces, first recall that by Cayley-Hamilton theorem

$$U^2 = \mathrm{Tr}(U) \cdot U - \det(U) I_2 \qquad (*).$$

Multiplying (*) by U^{n-2} and computing the trace of both sides we obtain a recurrence formula

$$\mathrm{Tr}(U^n) = \mathrm{Tr}(U).\mathrm{Tr}(U^{n-1}) - \det(U).\mathrm{Tr}(U^{n-2}) \qquad (**)$$

This shows that traces of powers may be expressed in terms of the trace and determinant of the initial matrix. If $\mathrm{Tr}(U) = 2$ and $\det(U) = 1$ then also $\mathrm{Tr}(U^n) = 2$.

There is a relationship between unipotents and the order of units in $GL_2(R)$. Indeed we can prove the following

Proposition 2 *If* $\mathrm{char}(R) = 0$ *then the only finite order unipotent in* $GL_2(R)$ *is* I_2.

Proof. Since we are looking for trace 2 and determinant 1 matrices we can start with a matrix $U = \begin{bmatrix} a+2 & b \\ c & -a \end{bmatrix}$ with $a(a+2) + bc = -1$. Repeatedly using this last equality we get $U^2 = \begin{bmatrix} (a+2)^2 + bc & 2b \\ 2c & a^2 + bc \end{bmatrix} = \begin{bmatrix} 2a+3 & 2b \\ 2c & -2a-1 \end{bmatrix}$ and $U^3 = \begin{bmatrix} 3a+4 & 3b \\ 3c & -3a-2 \end{bmatrix}$.

An easy induction shows (case $n = 2$ and $n = 3$ are displayed above) that for $U = \begin{bmatrix} a+2 & b \\ c & -a \end{bmatrix}$ and positive integer n we have $U^n = \begin{bmatrix} na + (n+1) & nb \\ nc & -na - (n-1) \end{bmatrix}$. Therefore $U^n = I_2$ iff $b = c = 0$ and $na + (n+1) = 1 = -na - (n-1)$ which happens iff $a = -1$. Hence $U^n = I_2$ iff $U = I_2$. ■

Alternative proof [Breaz]: Idempotents have the form $U = I_2 + N$ with $N^2 = 0_2$. Hence, using Newton's binomial, $U^n = (I_2 + N)^n = I_2 + nN$ for any positive integer n. Since this is $= I_2$ only for $N = 0_2$, the statement follows.

Examples. 1) $U_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is an order 2 unit, which is not unipotent.

2) $U_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This is an infinite order unit, which is unipotent.

$U_3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This is an infinite order unit, which is not unipotent.