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A NEW CLASS OF SEMIPRIME RINGS

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Abstract. A ring R is called unit-semiprime if for any $a \in R$, $a = 0$ whenever $aua = 0$ for all units $u \in U(R)$. This turns out to be a proper class of semiprime rings which, among others, includes the reduced rings and the unit-regular rings, is closed to matrix extensions but is not Morita invariant. The usual Ring theoretic constructions are investigated and connections with some other known classes of rings are established.

1. Introduction

In this note, we consider only nonzero (associative) rings with identity. The group of units of a ring R is denoted by $U(R)$ and the center of the ring R is denoted by $Z(R)$. For unexplained definitions and results in Ring Theory we refer to [5].

Especially in noncommutative rings, prime and semiprime are important classes of rings. Their definitions are (among others) given as follows: an ideal P of a ring R is *prime* if for any $a, b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$, and, *semiprime* if for any $a \in R$, $aRa \subseteq P$ implies $a \in P$. Further, a ring R is *(semi)prime* if (0) is a (semi)prime ideal in R.

Our starting definition is obtained by formally replacing the whole ring in the above definitions, by the group of units $U(R)$. Thus, an ideal P of a ring R is *unit-prime* if for any $a, b \in R$, $aU(R)b \subseteq P$ implies $a \in P$ or $b \in P$, and, *unit-semiprime* if for any $a \in R$, $aU(R)a \subseteq P$ implies $a \in P$. Recall that an ideal P is called *completely prime* if for any $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. Since

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$1 \in U(R)$, every completely prime ideal is unit-prime, that is, for ideals

$$\{\text{completely} - \text{prime}\} \subseteq \{\text{unit} - \text{prime}\} \subseteq \{\text{prime}\}$$

Since it is well-known that in a commutative ring every prime ideal is completely prime, *in the commutative case* all these classes of ideals coincide.

Further, a ring R is *unit-(semi)prime* if (0) is a unit-(semi)prime ideal of R.

The main goal of this paper is to investigate the class of unit-semiprime rings.

Reduced rings (e.g. rings with trivial group of units) and unit-regular rings (e.g. semisimple) are unit-semiprime. Moreover, we prove in Section 3 that von Neumann regular rings are also unit-semiprime. We have the following chart

$$\begin{array}{ccccccc} & & & & \text{prime} & & \\ & & & \nearrow & & \searrow & \\ \text{domain} & \longrightarrow & \text{unit} - \text{prime} & & & & \text{semiprime} \\ & & & \searrow & & \nearrow & \\ & & & & \text{unit} - \text{semiprime} & & \end{array}$$

None of these implications can be reversed and prime and unit-semiprime are independent properties. Suitable examples are given in the last section.

It is easy to see that in special conditions these new definitions coincide with the old ones. Such conditions are reversible rings (Cohn [2], i.e., $ab = 0$ implies $ba = 0$) or, rings additively generated by their units. In a special case, a commutative ring is unit-prime iff it is prime iff it is an integral domain, and, is unit-semiprime iff it is semiprime iff it is reduced.

In Section two, Ring theoretic constructions (i.e., products, quotients, polynomial rings, corners, representation as subdirect products) are studied, separating the matrix extensions in Section three, where we also conclude that unit-semiprime is not a Morita invariant property. In Section four some examples are given and connections with other classes of rings are made. To encourage future work some open questions are stated all over the paper.

2. Ring Theory constructions

By denial, a ring is unit-semiprime iff for every $a \neq 0$ there is a unit $u \in U(R)$ with $aua \neq 0$. Since for $a^2 \neq 0$, the condition if obviously fulfilled, the condition must be verified only for nonzero zero-square elements: R is unit-semiprime iff for every $a \neq 0$ with $a^2 = 0$, there is a unit $u \in U(R)$ with $aua \neq 0$.

Since intersections of unit-semiprime ideals are unit-semiprime, for an ideal A, consider $\overline{A} = \bigcap\{P | P$ is unit-semiprime ideal in R, $A \subseteq P\}$. This is the smallest unit-semiprime ideal which includes A. Thus we immediately obtain

Proposition 1. *The set of all unit-semiprime ideals of a ring, ordered by inclusion, forms a complete lattice.*

Proof. As usual, for an arbitrary family $\{P_i\}_{i\in I}$ of unit-semiprime ideals, the inf is the intersection $\bigcap\limits_{i\in I} P_i$ and the sup is $\overline{\bigcup\limits_{i\in I} P_i}$. $\square$

Proposition 2. *A product of rings is unit-semiprime iff each component is unit-semiprime.*

Proof. To simplify the writing we prove this for two rings R, S. Suppose $R\times S$ is unit-semiprime and let $0\neq r$, $r^2=0$ so that $(0,0)\neq(r,0)$ and $(r,0)^2=(0,0)$. Since $U(R\times S)=U(R)\times U(S)$ there are $u\in U(R)$ and $v\in U(S)$ such that $(r,0)(u,v)(r,0)=(rur,0)\neq(0,0)$, as desired. Conversely, assume $(0,0)\neq(r,s)\in R\times S$ and $(r,s)^2=(r^2,s^2)=(0,0)$. Further, suppose $r\neq 0$ (the case $s\neq 0$ is symmetric). Thus, since $r^2=0$, there is $u\in U(R)$ with $rur\neq 0$. Hence $(r,s)(u,1)(r,s)=(rur,s^2)\neq(0,0)$ as claimed. $\square$

Next recall the following

Lemma 3. *Let $f:R\longrightarrow R'$ be a ring homomorphism (of rings with identities).*

(i) If f is surjective then f is unital (i.e., $f(1)=1'$).

(ii) If f is unital then $f(U(R))\leq U(R')$.

(iii) Denote $\overset{-1}{f}(X')=\{r\in R|f(r)\in X'\}$ the inverse image of X' by f. If P is an ideal of R then $f(P)$ is an ideal of $f(R)$. If P' is an ideal in R' then $\overset{-1}{f}(P')$ is an ideal of R.

(iv) If A is a subring of R then $\overset{-1}{f}(f(A))=A+\ker f$. Hence if $\ker f\leq A$ then $\overset{-1}{f}(f(A))=A$.

Thus we can prove a correspondence result for unit-semiprime ideals.

Theorem 4. *Let $f:R\longrightarrow R'$ be a surjective ring homomorphism. Then*

(a) if P is a unit-semiprime ideal of R and $\ker f\leq P$ then $f(P)$ is unit-semiprime in R'.

(b) if P' is a unit-semiprime ideal of R' and $f(U(R))=U(R')$ then $\overset{-1}{f}(P')$ is unit-semiprime in R.

Proof. As seen in the previous Lemma, both correspondences preserve ideals.

(a) First notice that for any subset S of R and ideal P of R, since $\ker f\leq P$, $f(S)\subseteq f(P)$ implies $S\subseteq P$. Indeed, $f(S)\subseteq f(P)$ implies $S\subseteq\overset{-1}{f}(f(S))\subseteq$

$\overset{-1}{f}(f(P)) = P + \ker f = P$ (the first inclusion, always true). Next, suppose $a'U(R')a' \subseteq f(P)$ for some $a' \in R'$. Since $f(U(R)) \leq U(R')$ and f is surjective we have $f(aU(R)a) \subseteq f(P)$ for some $a \in R$. But then $aU(R)a) \subseteq P$ from the first part and $a \in P$ by hypothesis. Finally, $a' = f(a) \in f(P)$ as desired.

(b) Suppose $aU(R)a \subseteq \overset{-1}{f}(P')$. Then $f(aU(R)a) = f(a)f(U(R))f(a) = f(a)U(R')f(a) \subseteq f(\overset{-1}{f}(P')) = P'$ and so $f(a) \in P'$ by hypothesis. Hence $a \in \overset{-1}{f}(P')$, as required. □

Remark. Obviously $f(U(R)) \subseteq U(R')$ holds for every unital ring homomorphism. To have the best possible correspondence between unit-semiprime ideals we need equality. For factor rings, if $\pi_A : R \longrightarrow R/A$, $\pi_A(r) = r + A$ denotes the canonical projection (surjective and $\ker \pi_A = A$), the corresponding equality $\pi_A(U(R)) = \{u + A | u \in U(R)\} = U(R/A)$ amounts to lifting units, a notion (see [6] for references) defined similarly with the well-known lifting of idempotents.

Definition. If A is an ideal of a ring R, we say *a unit $x \in R/A$ can be lifted* to R if there exists a unit $u \in U(R)$ such that $\pi_A(u) = x$ (i.e., $u + A = x$).

Corollary 5. *Let A be an ideal of the ring R such that units in R/A can be lifted to R. An ideal of the quotient ring R/A is unit-semiprime iff it has the form P/A with P a unit-semiprime ideal of R which includes A.*

Proof. Just apply the previous Theorem to the canonical projection π_A. Indeed, by (a), if P is unit-semiprime and $A \leq P$ then P/A is unit-semiprime in R/A and, by (b), since units can be lifted, every unit-semiprime ideal P' of R/A has the form P/A (with $A \leq P = \overset{-1}{\pi_A}(P')$ and $P' = P/A$). □

Corollary 6. *If for an ideal P of a ring R, units in R/P can be lifted to R, then the factor ring R/P is unit-semiprime iff the ideal P is unit-semiprime.*

Proof. From the previous Corollary. □

Clearly, proving results on unit-semiprime polynomial rings depends to what extent we know the invertible polynomials. An easy example is: for any integral domain D, the polynomial ring $D[X]$ is unit-semiprime.

Since for commutative rings, unit-semiprime, semiprime and reduced are equivalent conditions, we obtain at once (see **(10.18)** in [5])

Theorem 7. *For a set T of indeterminates, the polynomial ring $R[T]$ over a commutative (unital) ring is unit-semiprime iff R is reduced.*

A similar result holds for unit-prime rings, that is

Theorem 8. *For a set T of indeterminates, the polynomial ring $R[T]$ over a commutative (unital) ring is unit-prime iff R is an (integral) domain.*

A well-known fact is that semiprime rings are characterized by being isomorphic to subdirect products of prime rings. Since unit-semiprime rings are semiprime, these are also isomorphic to subdirect products of prime rings. However, we may wonder whether these are isomorphic to subdirect products of unit-prime rings. For this we need the following

Claim 9. *For an ideal P the following are equivalent:*
(1) P is unit-semiprime,
(2) P is an intersection of unit-prime ideals.
(3) P is the intersection of all the unit-prime ideals containing P.

For semiprime rings this holds because the prime radical of an ideal P equals the intersection of all prime ideals containing P. For unit-(semi)prime, clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1) hold, but (1) $\Rightarrow$ (3) generally fails. The problem here are the maximal ideals: very likely, but we were not able to give an example, these need not be unit-prime.

Therefore we state the following

Question. Are maximal ideals *unit-prime* ? Or at least unit-semiprime?

3. Matrix rings

First it is easy to discard rings of (upper) triangular matrices. Since these are not even semiprime, the ring of triangular matrices over any (nonzero) ring is not unit-semiprime.

Clearly, *trivial extensions* $T(R, M)$ for any nonzero ring R and R-R-bimodule M are not unit-semiprime. Moreover, (formal) triangular rings, i.e., for two rings R, S and an R-S-bimodule M, the rings $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, are not unit-semiprime.

Further, since by a result of Henriksen, matrix rings are additively generated by units (see [3], Theorem 3), for matrix rings, $\mathcal{M}_n(R)$ is unit-semiprime iff $\mathcal{M}_n(R)$ is semiprime. This way, if R is a semiprime ring, $\mathcal{M}_n(R)$ is not only semiprime but also unit-semiprime.

To make this paper self-contained (the proof of Henriksen's theorem is one page long) in the sequel we give a direct proof for the following

Theorem 10. *Matrix rings over unit-semiprime rings are unit-semiprime.*

Proof. The proof will be by induction on n, the case $n=1$ being trivial. For $n \geq 2$, suppose that $\mathcal{M}_{n-1}(R)$ is unit-semiprime and take $A = \begin{bmatrix} M & \beta \\ \alpha & d \end{bmatrix} \in \mathcal{M}_n(R)$ with $M \in \mathcal{M}_{n-1}(R)$, α an $n-1$ row, β an $n-1$ column and $d \in R$, such that $A \neq 0$ but $A^2 = 0$. By block multiplication $A^2 = \begin{bmatrix} M^2+\beta\alpha & M\beta+\beta d \\ \alpha M + d\alpha & \alpha\beta + d^2 \end{bmatrix} = 0$ and we go into several cases.

Case 1. $\alpha = 0$ and so $M^2 = 0_{n-1}$ and $d^2 = 0$.

(1) $M = 0_{n-1}$ and $d = 0$. Here $A = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ with nonzero column $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \neq 0$, say $\beta_i \neq 0$. We use the $n-1$ row γ with all entries zero excepting the i-th entry which we denote by t and the invertible matrix $U = \begin{bmatrix} I_{n-1} & 0 \\ \gamma & 1 \end{bmatrix}$.

Since $\gamma\beta = t\beta_i$ we obtain

$$AUA = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta\gamma\beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta(t\beta_i) \\ 0 & 0 \end{bmatrix}.$$

Therefore, if $\beta_i^2 \neq 0$ we can take $t = 1$ and if $\beta_i^2 = 0$, since R is unit-semiprime, there exists a unit $u \in U(R)$ such that $\beta_i u \beta_i \neq 0$, and we can take $t = u$. In both cases $AUA \neq 0_n$, as desired.

(2) $M = 0_{n-1}$ and $d \neq 0$. Since $d^2 = 0$, there is a unit $u \in U(R)$ with $dud \neq 0$ and for the invertible matrix $U = \begin{bmatrix} I_{n-1} & 0 \\ 0 & u \end{bmatrix}$ we get

$$AUA = \begin{bmatrix} 0 & \beta \\ 0 & d \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & \beta ud \\ 0 & dud \end{bmatrix} \neq 0_n.$$

(3) $M \neq 0_{n-1}$. By induction hypothesis, there exists an invertible $(n-1) \times (n-1)$ matrix V such that $MVM \neq 0_{n-1}$. Then for the invertible matrix $U = \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}$ we obtain

$$AUA = \begin{bmatrix} M & \beta \\ 0 & d \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M & \beta \\ 0 & d \end{bmatrix} = \begin{bmatrix} MVM & MV\beta+\beta d \\ 0 & d^2 \end{bmatrix} \neq 0_n.$$

Case 2. The row $\alpha = \begin{bmatrix} \alpha_1 & \ldots & \alpha_{n-1} \end{bmatrix} \neq 0$, say $\alpha_j \neq 0$. Now we use the column δ with all entries zero excepting the j-th entry denoted s. Then $\alpha\delta = \alpha_j s$, we take the invertible matrix $U = \begin{bmatrix} I_{n-1} & \delta \\ 0 & 1 \end{bmatrix}$ and the following computation (we use the vanishing of all entries in the initial block multiplication $A^2 = 0_n$) solves this final case

$AUA = \begin{bmatrix} M & \beta \\ \alpha & d \end{bmatrix} \begin{bmatrix} I_{n-1} & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M & \beta \\ \alpha & d \end{bmatrix} = \begin{bmatrix} M\delta\alpha & M\delta d \\ \alpha\delta\alpha & \alpha\delta d \end{bmatrix} \neq 0_n$. Indeed, if $\alpha_j^2 \neq 0$ we can take $s = 1$ and if $\alpha_j^2 = 0$ we can take $s = u$ with a unit u given by the unit-semiprime ring for $\alpha_j u \alpha_j \neq 0$, and the proof is complete. □

In the last section, an example of prime ring which is not unit-semiprime (constructed by G. Bergman) will be given. Using our remarks before the proposition above, we can justify the following

Proposition 11. *The unit-semiprime property fails to pass to corners.*

Proof. (Jerzy Matczuk) Take S to be any prime ring which is not unit-semiprime. Then $\mathcal{M}_2(S)$ is a prime ring, so unit-prime since it is (additively) generated by units, and finally unit-semiprime. However its corner S is not unit-semiprime. □

Therefore

Corollary 12. *Unit-semiprime is not a Morita invariant property of rings.*

Since unit-regular rings are unit-semiprime, in particular semisimple and simple Artinian rings are unit-semiprime. Therefore

Proposition 13. *For any division ring D and positive integer n, the matrix ring $\mathcal{M}_n(D)$ is unit-semiprime.*

Thus we have an analogue for **(10.24)** from [5]

Corollary 14. *A ring is semisimple iff it is unit-semiprime and left Artinian.*

Notice that for matrix rings over commutative rings the converse is obvious: if $\mathcal{M}_n(R)$ is unit-semiprime, it is also semiprime and so (see **(10.20)** in [5]), R is semiprime. But if R is also commutative, it is also unit-semiprime.

Von Neumann regular rings are clearly semiprime. Moreover, as already mentioned in the Introduction we can prove [1] the following

Proposition 15. *Von Neumann regular rings are unit-semiprime.*

Proof. Take an element $0 \neq a \in R$ with $a^2 = 0$. By hypothesis there exists an element $x \in R$ with $a = axa$. Then $e = xa$ is an idempotent and for any $r \in R$, $1 + er\overline{e}$ is a unit. Choose $r = x$, that is $u = 1 + xax(1 - xa) \in U(R)$, and compute $aua = a[1 + xax(1 - xa)]a = axa = a \neq 0$, since $a^2 = 0$. □

[1] After completing this paper it was brought to my attention that more can be proved: any nilpotent element in a regular ring is unit-regular.

Here is the right place to mention an example which was studied and used a lot, related to unit-regular, regular or Dedekind finite rings: $R = \mathrm{End}_D(V)$, the endomorphism ring of an infinite dimensional vector space V over a division ring D. Since this is a regular ring (which is not unit-regular), R is unit-semiprime. Here is a short undergraduate

Proof. Let $0 \neq a \in R$ with $a^2 = 0$. Then $0 < \mathrm{im}a \leq \ker a < V$, we choose a basis B for $\mathrm{im}a$, extend it to $\ker a$ (say $B \cup B'$), and to the whole V (say $B \cup B' \cup B''$). Thus for every $v \in B''$, $a(v) \neq 0$. Select elements $b \in B$, $b'' \in B''$ and any automorphism (change of basis) u which maps b into b''. If $x \in V$ is such that $b = a(x)$ then $aua(x) = a(b'') \neq 0$. $\square$

Yet another proof can be given noticing that if $\dim_D(V) \geq 2$ then R is generated by its units (Zelinsky [7] 1954).

Due to known properties of the above endomorphism ring, we conclude that unit-semiprime rings need not be Dedekind finite, and the above is an example of unit-semiprime ring which is not unit-regular.

Since the last two results show that semisimple rings and Von Neumann regular rings are unit-semiprime, the following is naturally in order:

Question. *Are semiprimitive (i.e., J-semisimple) rings, unit-semiprime*?

Recall that semiprimitive rings are semiprime and notice that, according to **(5.2)** [5] and Theorem 7, $R[T]$ is *unit-semiprime* iff it is *semiprimitive* iff R is reduced. So these are equivalent properties for polynomial rings (over commutative rings).

4. Examples, connections

We first refer to the chart given in the Introduction and show that *"prime" and "unit-semiprime" are independent properties.*

In one direction, since reduced rings are not always prime, but are unit-semiprime, a unit-semiprime ring might not be prime.

Conversely, the following is an example of *a prime ring* (and so semiprime too) *which is not unit-semiprime*, given by George Bergman.

Consider $R = k\left\langle x, y | x^2 = 0\right\rangle$, for k any field. R is prime (and so also semiprime), since for any two nonzero elements $r, s \in R$ we have $rys \neq 0$.

To show that it is not unit-semiprime, we need to know its group of units. To do this, regard R as the coproduct over k of $R_1 = k[x|x^2 = 0]$ and $R_2 = k[y]$. Then from Corollary 2.16 of [1], one can deduce that the units of R are just the elements $c + dx + xrx$, where $c, d \in k$ and $r \in R$. We see that for any such unit, we have $x(c + dx + xrx)x = 0$; so unit-semiprimeness fails.

To show that the arrows in the chart are not reversible, just notice that any commutative reduced ring which is not an integral domain is (unit-)semiprime but not (unit-)prime.

Related to the left arrow of this chart we prove

Proposition 16. *Let R be a ring. Then R is a domain if and only if R is unit-prime and units commute with nilpotent elements.*

PROOF. The conditions are clearly necessary. Conversely, assume R is a unit-prime and units commute with nilpotent elements. Let $a, b \in R$ with $ab = 0$. Then $rab = 0$ for all $r \in R$. Since $(bra)^2 = 0$, bra commutes with units. Let $u \in U(R)$ and $aubraub \in (aub)R(aub)$ for any $r \in R$. Hence $a\underline{ubra}ub = \underline{ab}rau^2b = 0$, and so $(aub)R(aub) = 0$. Since R is (semi)prime, we have $aub = 0$ for all $u \in U(R)$ and so $aU(R)b = 0$. By unit-primeness we get $a = 0$ or $b = 0$, and R is a domain. $\square$

We also have

Proposition 17. *Let R be a ring. Then R is reduced if and only if R is unit-semiprime and units commute with nilpotent elements.*

PROOF. Only one way needs verification. Assume $a \in R$ with $a^2 = 0$. Then a commutes with units and since R is unit-semiprime, $aU(R)a = a^2U(R) = 0$, it follows that $a = 0$, and so R is reduced. $\square$

Rings whose units commute with nilpotent elements will be called *uni* rings and studied elsewhere.

Recall that a ring is *unit-central* (see [4]) if $U(R) \subseteq Z(R)$. Then

Proposition 18. *If a unit-semiprime ring R is unit-central, or, has central zerosquare elements, then R is reduced.*

PROOF. Obvious. $\square$

Actually even less suffices: for every $0 \neq a \in R$, $aU(R) \subseteq Ra$, or, $U(R)a \subseteq aR$. In particular, this holds for left or right duo rings.

It is known that unit-regular rings are clean and these are exchange. However, not even commutative unit-semiprime rings are clean or exchange: $\mathbf{Z}$ is a domain and so unit-semiprime, but not exchange (and so not clean). For a non-commutative (but Abelian) example we can take the ring of integral matrices $R = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d (\mathrm{mod} 2), b \equiv c \equiv 0 (\mathrm{mod} 2) \}$.

Proof. Since (by computation) the nonzero zero-square matrices are $N_1 = \begin{bmatrix} 0 & 2b \\ 0 & 0 \end{bmatrix}$, $b \in \mathbf{Z}$, or, $N_2 = \begin{bmatrix} 0 & 0 \\ 2c & 0 \end{bmatrix}$, $c \in \mathbf{Z}$, or $N_3 = 2\begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix}$ with $a, b \in \mathbf{Z}^*$ and b a divisor of a^2, the unit $U = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ suits well for $NUN \neq 0_2$ in all cases. While this is easily checked for N_1 or N_2, for N_3 observe that $N_3UN_3 = \begin{bmatrix} * & 2(b^2 - a^2 - ab) \\ * & * \end{bmatrix} \neq 0_2$ over the integers (indeed, $x^2 + x - 1 = 0$ has no rational solutions). This ring is Abelian, has only trivial idempotents and so is not clean nor exchange.

In a local ring the unique maximal ideal $R - U(R) = J(R)$ is also unit-semiprime: for $a \notin J(A)$ means $a \in U(A)$ and so $aU(R)a \subseteq U(R)$ and $aU(R)a \nsubseteq J(A)$.

However, *local rings need not be unit-semiprime*: by (**19.8**) [5], let k be a division ring, R the ring of upper triangular $n \times n$ matrices over k and A the subring of R consisting of matrices with a constant diagonal. Then A is *local* and $J(A) = J(R) = \{$matrices with zero diagonal$\}$ is the unique maximal ideal. As seen in the beginning of Section three, R and similarly A are *not unit-semiprime.*

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