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TORSION IN LATTICES

by

GRIGORE G. CĂLUGĂREANU

We will work within a complete lattice L, with smallest and largest elements denoted by 0 resp. 1.

1. Terminology. An element a is *essential* in L if $a \wedge c \neq 0$ holds for every $c \neq 0$ in L. An element c is a *pseudocomplement* of an element b in L if $b \wedge c = 0$ and c is maximal with this property. The lattice L is *pseudocomplemented* if for every interval $I \subseteq L$ and every $b \in I$, there is a pseudocomplement of a in I. For $a \leqslant b$ elements in L we shall use the notation $b/a = \{x \in L / a \leqslant x \leqslant b\}$. A non-zero element a of L is an *atom* if $b < a$ implies $b = 0$. The *socle* $s(L)$ of L is the join of all atoms of L.

A lattice L is *noetherian/artinian/*if L satisfies the ascending chain (or the maximum) condition/resp. the descending chain (or the minimum) condition/. L will be called *atomic* if for every non-zero element a, the sublattice $a/0$ contains atoms and *strongly atomic* if for every $a < b$, the sublattice b/a contains atoms.

L will be called *torsion* if for every $a \neq 1$, the sublattice $1/a$ contains atoms.

2. Well-known results.

(A) The essential elements form a filter in L.

(B) If L is modular and compactly generated, the socle $s(L)$ is equal to the meet of all essential elements of L.

(C) If L is atomic the socle $s(L)$ is essential. The filter mentioned in (A) is then principal, generated by $s(L)$.

(D) If L is modular, c is a pseudocomplement of b iff $b \wedge c = 0$ and $b \vee c$ is essential in $1/c$.

(E) L is strongly atomic iff $b \neq 1$ implies $1/b$ atomic.

(F) Every artinian lattice is strongly atomic. Every strongly atomic lattice is torsion.

(G) Every noetherian torsion lattice is artinian.

3. Results concerning torsion.

PROPOSITION 1 *Every modular pseudocomplemented torsion lattice is atomic.*

Proof. Let a be a non-zero element in L and b be a pseudocomplement of a. According to (D), $a \vee b$ is essential in $1/b$ and $a \wedge b = 0$.

Therefore $b \neq 1$ (otherwise $a = 0$) and L being torsion, $1/b$ contains atoms. If c is an atom in $1/b$, $(a \vee b) \wedge c \neq b$ implies $(a \vee b) \wedge c = b$ and then $c \in (a \vee b/b)$. Using a well-known isomorphism theorem in modular lattices we have $a \vee b/b \cong a/a \wedge b = a/0$. Hence $a \wedge c$ is atom in $a/0$ and L is atomic.

Remark. Every superior continuous modular lattice is pseudocomplemented. Hence, for such lattices the following implications are valid: artinian $\Rightarrow$ strongly atomic $\Rightarrow$ torsion $\Rightarrow$ atomic.

PROPOSITION 2 *Let a be an element of a modular pseudocomplemented lattice. Then L is torsion iff both sublattices $a/0$ and $1/a$ are torsion.*

Proof. L torsion implies $1/a$ torsion is clear. If $b < a$ then $a/b \subseteq 1/b$. The last sublatice being torsion, we use prop. 1 in order to find atoms in a/b. Conversely, let $b \neq 1$ be element in L. If $b \wedge a = a$ then $b \in 1/a$ and $1/a$ being torsion $1/b$ contains atoms. If $b \wedge a \neq a$ then $a/0$ torsion implies $a/a \wedge b$ contains atoms. The rest is now done by the above mentioned isomorphism.

Definition. An element a of L is called *torsion* if the interval $a/0$ is torsion. We shall denote by $T(L)$ the set of the torsion elements of L. Trivially, $T(L)$ contains al l the atoms of L and 0.

Remark. If L is a modular pseudocomplemented lattice we easily recover the well-known properties for a hereditary simple torsion theory.

($T1$) If $b \leqslant a$ and $a \in T(L)$ then $b \in T(L)$ and a/b is torsion.

($T2$) Conversely, if $b \in T(L)$, $a \geqslant b$ and a/b is torsion then $a \in T(L)$.

($T3$) If $\{a_i\}_{i \in I} \subseteq T(L)$ then $\bigvee_{i \in I} a_i \in T(L)$.

As for ($T3$), if $0 \leqslant c < \bigvee_{i \in I} a_i$ there is a $j \in I$ such that $a_j \not\leqslant c$. In this case $a_j \wedge c < a_j$ and a_j being torsion $a_j/a_j \wedge c$ contains atoms. But then $a_j \wedge c/c$ and hence $\bigvee a_i/c$ contains atoms.

Definition. The *torsion part* of a lattice L, denoted $t(L)$ is the join of all the torsion elements of L.

PROPOSITION 3. *If L is a complete modular lattice then $T(L)$ is a principal ideal, generated by $t(L)$.*

Easy consequence of ($T3$).

PROPOSITION 4 *$s(L) \leqslant t(L)$ and $s(L) = 0$ iff $t(L) = 0$.*

Clear.

PROPOSITION 5. *The torsion part has the "radical" property, i.e. $t(1/t(L)) = t(L)$.*

Proof. We show that $s(1/t(L)) = t(L)$. Let b be an atom in $1/t(L)$. We shall prove that $b \in T(L)$, which will be a contradiction.

Let a be such that $b > a$. If $a = t(L)$ then b is atom in b/a and if $a < t(L)$, $t(L)/a$ and consequently b/a contains atoms. In the remaining case, $a \not\leqslant t(L)$, we have $t(L) < a \vee t(L) \leqslant b$ and hence $a \vee t(L) = b$. Again, using the isomorphism theorem $b/a = a \vee t(L)/a \cong t(L)/a \wedge t(L)$ and all these intervals contain atoms.

Remark. Moreover, if for an element c of L we have $t(1/c) = c$ then $t(L) \leqslant c$.

Proof. If there is an element $a \in T(L)$ such that $a \not< c$, then $a \wedge \wedge c < a$ and $c < a \vee c$ so that $a/a \wedge c \subseteq a/0$ and $a \vee c/c \subseteq 1/c$ and all these intervals contain atoms. Hence $s(1/c) \neq c$ and $t(1/c) \neq c$.

Definition. The lattice L satisfies the *restricted socle condition* (RSC) if $b \neq 1$, b esential in L implies $1/b$ contains atoms ($s(1/b) \neq b$).

PROPOSITION 6 *A modular atomic lattice is torsion iff it is RSC.*

Proof. Only one way needs comments. We prove that if L is atomic and RSC then L is torsion. For an element $b \neq 1$ of L only two cases are possible: b essential in L or there is an non-zero element a of L such that $a \wedge b = 0$. In the first case, RSC assures that $1/b$ contains atoms and in the second one we have $a/0 = a/a \wedge b \cong a \vee b/b$. L being atomic all these contain atoms and so is also $1/b$. Hence L is torsion.

PROPOSITION 7 *If L is a compactly generated modular lattice with RSC then the following properties are equivalent:*

(*i*) *s is essential in L*; (*ii*) *$s(L) \leqslant s$ and $1/s$ is torsion.*

Proof. If s is essential in L it is known from [3] that $s(L) \leqslant s$. Further if $s \leqslant a < 1$ then a is essential in L and RSC implies that $1/a$ contains atoms. Hence $1/s$ is torsion.

Conversely, let s satisfy (*ii*) but not (*i*). There is $a \neq 0$ such that $s \wedge a = 0$. We again have $a/0 \cong s \vee a/s$ and $1/s$ being atomic, let a' be atom in $a/0$. The conditions $a' \leqslant a$, $s(L) \leqslant s$ and $a \wedge s = 0$ imply that $a' \wedge s(L) = 0$. But then $a' \not< s(L)$, which is a contradiction.

Corollary. *If L is a compactly generated modular lattice with RSC then the following properties are equivalent:*

(*a*) *c is a pseudocomplement of b in L*; (*b*) *$b \vee c$ is essential in $1/c$*; (*c*) *$s(1/c) \leqslant b \vee c$ and $1/b \vee c$ is torsion.*

Consequently, if M is a left module over the ring R, and M satisfies RSC for submodules, the submodule C is a pseudocomplement of the submodule B iff $S(M/C) \leqslant B \oplus C$ and $M/B \oplus C$ is semiartinian.

Remark. For Z-modules, i.e. abelian groups the restricted socle condition is always true, and the conditions mentioned above (semiartinian = = torsion) caracterize pseudocomplements (or B-high subgroups) of abelian groups.

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Faculty of Mathematics
,,Babeș-Bolyai" University
3400 Cluj-Napoca