TORSION-FREE COMPONENTS AND TOPOLOGY

by

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All over this paper "group" means abelian group. For a group A, $T(A)$ denotes the torsion part, A is mixed if $0 \neq A \neq T(A)$ and A is splitting if $T(A)$ is a direct summand of A. If A is a non--splitting mixed group, A cannot be recaptured from $T(A)$ and $A/T(A)$.

We recall the following notion: if B and C are subgroups of a group A, C is called B-high (or a pseudocomplement of B) if C is maximal with the property $B \cap C = 0$. This notion generalizes the direct sum (if $A = B \oplus C$ then C is B-high and B is C-high, so that B and C are mutually high).

In a (non-splitting mixed) group A we consider $T(A)$-high subgroups. All these are torsion-free (maximal among the torsion-free subgroups) and generally form an infinite set. We call them torsion--free components.

The torsion-free components are neat subgroups, none of them being pure if the group is a non-splitting mixed one. Their ranks are all equal to the torsion-free rank of the group. If A is mixed the set of the torsion-free components is infinite. The sum of all these components is the whole group and their intersection is zero.

The torsion part $T(A)$ and the torsion-free components are "close" enough: if F is a torsion-free components, by definition F is $T(A)$-high but in this case $T(A)$ is also F-high.

The author studies now the following problems: characterize the non-splitting mixed groups which have a free component; characterize the non-splitting mixed groups A which have two components of sum A.

In a non-splitting mixed group one tries to make some "order" in the infinite set of all the torsion-free components. A subgroup F is such a component iff F is torsion-free, neat and A/F is torsion. So, one considers the set Π of all the subgroups U of the group A such that A/U is torsion. In this way, we reach the second subject.

The set Π is a filter in the lattice $L(A)$ or all the subgroups of a group A and so it defines a linear topology on A which we call the Π-topology of A.

This is a functorial topology (i.e. defines a functor from Ab to Abtop, i.e. group morphisms are continuous in the Π-topology) and an ideal one (every epimorphism is open). This topology has the following properties (according to my knowledge this topology was not studied):

1) The Π-topology is Hausdorf; for torsion groups this is the discrete topology.
2) The Π-topology containes all the essential subgroups of A (B is essential in A iff $S(A)$ B and A/B is torsion; $S(A)$ denotes the socle of the group A).
3) In the Π-topology of a group A the only dense subgroup is A.
4) A subgroup B of A is called Π-concordant if the Π-topology of B is equal to the relative Π-topology of B in A. Then, B is Π-concordant in A iff A/B is torsion.
5) The functor defined by Π commutes with direct sums.
6) Every mixed group is complete in his Π-topology.
7) The Π-topology is not completable (see $\mathbb{Z}$ and his completion $\prod_{p\in\mathbb{R}} J_p$).

We finally record another open problems:

P3: characterize the torsion-free groups which are complete in their Π-topology.

P4: which subgroups are closed in the Π-topology of a group?

P5: characterize the groups in which the $\mathbb{Z}$-adic topology and the Π-topology are equal (e.g. $\mathbb{Z}$, the integers).

REFERENCES

. A. Mader, "Basic concepts of functorial topologies", Springer Lecture Notes 874, 1981.