Subrings of matrix rings

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Abstract

When comparing several classes of rings, matrix examples are frequently used. In the sequel, we intend to put some order in this matter.

For a positive integer $n \geq 2$ and a nonzero commutative ring with identity R, denote by $S = \mathcal{M}_n(R)$ the (full) matrix ring (or unital R-algebra). Our goal is to classify some of the subrings of S, and among these, subrings with identity and commutative subrings, respectively. Actually these will be subalgebras and will be described by determining subrings among finitely generated free R-submodules.

What follows is a natural way of presenting such subrings.

We denote by E_{ij} and call *matric unit*, the $n \times n$ matrix which has all entries zero, excepting the entry on the i-th row and j-th column, which is 1 (the name is not standard; we just want somehow to emphasize that these matrices are not ring units, i.e., invertible matrices). Notice that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Then obviously any matrix decomposes as a linear combination of all the n^2 matric units as follows $A = [a_{ij}] = \sum\limits_{i,j=1}^{n} a_{ij}E_{ij}$. Actually, it is easy to see that $S = \mathcal{M}_n(R)$ is a free left (or right) R-module on the n^2 element basis $\{E_{ij}\}$ (that is, these form a linearly independent generating set). As finitely generated free R-modules, $\mathcal{M}_n(R) \cong R^{n^2}$ are isomorphic R-algebras.

Since nonzero commutative rings have IBN (Invariant Basis Number, see [1]), any two bases on a finitely generated free module have the same (finite) number of elements.

1 Subalgebras generated by matric units

It is easy to determine the subrings among the free submodules generated by some matric units. Consider $N = \{1, 2, ..., n\}$ and for a binary relation $\rho \subseteq N \times N$, we consider the R-submodule M_ρ generated by $\{E_{ij} : (i,j) \in \rho\}$. Since there is no problem with the additive subgroup, in order to have a subring, only closure under multiplication is needed. Owing to the above multiplication rules for matric units, we obtain

Lemma 1 *M_ρ is a subring if and only if ρ is transitive (i.e., $\rho \circ \rho \subseteq \rho$).*

Proof. The condition is clearly sufficient. Conversely, let (i, j) and (j, k) be two pairs in ρ. Since M_ρ consists in linear combinations of E_{ij}'s with $(i, j) \in \rho$, both E_{ij} and E_{jk} belong to M_ρ. Hence so is their product $E_{ik} = E_{ij}E_{jk}$ and $(i, k) \in \rho$, as desired. ■

Similarly

Lemma 2 *M_ρ is a subring with identity if and only if ρ is a preorder (i.e., reflexive and transitive).*

Proof. M_ρ is a subring with identity iff $I_n = E_{11} + E_{22} + ... + E_{nn} \in M_\rho$, and this happens iff all $E_{ii} \in M_\rho$. ■

A well-known example is the natural (total) order on N given by $(i, j) \in \rho$ iff $i \leq j$: this way we obtain the subring with identity of all the upper triangular matrices.

For a commutative ring R with identity, $\left\{ \begin{bmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in R \right\}$ is, in $\mathcal{M}_3(R)$, the R-submodule generated by $\{E_{11}, E_{12}, E_{21}\}$. It is not subring and does not contain the identity (matrix I_3).

It is easy to give examples of idempotents and zero square elements generated by matric units: $E_{ii} + E_{ik}$, $k \neq i$ are idempotents and $E_{ii} + E_{ik} + E_{ki} + E_{kk}$, $k \neq i$ are idempotents. Moreover $E_{ii} + \sigma$ are idempotents, where σ means any sum of matric units on the i-th row (or column), outside the diagonal. Actually such σ's are zero square and $E_{ii}\sigma = \sigma$, $\sigma E_{ii} = 0$.

2 Subalgebras generated by linear combinations of matric units

When it comes to free submodules generated by some *given linearly independent linear combinations* of matric units, a description is somehow similar. These are also (finitely generated) free subalgebras of $\mathcal{M}_n(R) \cong R^{n^2}$.

Linear combinations of matric units are also determined by binary relations in $N \times N$, just listing the pairs from the lower script. For instance, $E_{11} + E_{12} + E_{21} + E_{22}$ is determined by $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

If $B = \{\rho_k\}_{k=1}^{s} \subseteq \mathcal{P}(N \times N)$ determine some generators, then the resulting R-submodule is a subring iff B is closed under composition: $\rho_k \circ \rho_l \in B \cup \varnothing$ for every $1 \leq k, l \leq s$.

Recall that $(\mathcal{P}(N \times N), \circ, \varnothing)$ is a semigroup with zero (and a complete lattice with respect to $\subseteq$). Thus

Proposition 3 *For a subset $B \subseteq \mathcal{P}(N \times N)$, the R-submodule generated as above is a subring if and only if B is a subsemigroup (with zero) of $(\mathcal{P}(N \times N), \circ, \varnothing)$.*

Proof. Indeed, all we need is $\rho, \tau \in B$ implies $\rho \circ \tau \in B$ or $= \varnothing$. ■

As for subrings with identity, if we denote Δ_M the equality relation on a set M, we have

Proposition 4 *For a subsemigroup (with zero) B of $(\mathcal{P}(N \times N), \circ, \varnothing)$, the R-submodule generated as above is a subring with identity if and only if B contains a partition of $\Delta_{N \times N}$. [This amounts to: there is a partition $\pi = B_1 \cup ... \cup B_m$ of $N = \{1, 2, ..., n\}$ such that all $\Delta_{B_j} \in B$].*

Proof. Indeed, only for such linear combinations $a_1 \sum_{i \in B_1} E_{ii} + ... + a_m \sum_{i \in B_m} E_{ii}$, we recapture the identity $I_n = E_{11} + E_{22} + ... + E_{nn}$, by taking $a_1 = a_2 = ... = a_m = 1$. ■

As an example, for $n = 3$, consider the set of matrices $\left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in R \right\}$. Such matrices can be presented as linear combinations $a(E_{11} + E_{22} + E_{33}) + bE_{12} + cE_{13} + dE_{23}$ (here $I_3 = E_{11} + E_{22} + E_{33}$). Therefore, this is a subalgebra generated by $E_{11} + E_{22} + E_{33}, E_{12}, E_{13}, E_{23}$, in the subalgebra of all the upper triangular matrices (which is generated by all $E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}$) (in order to check it is also a subring, just use the previous Propositions).

3 Special remarks

In the sequel, to simplify the wording, matric units E_{ij} with $i \neq j$ will be called *outside the diagonal*. Sometimes it will be useful to decompose a matrix $A = \sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{i \neq j} a_{ij} E_{ij}$, that is the diagonal and outside the diagonal (and $i < j$ for upper triangular matrices).

Further, in presenting this way a subring (with identity), two or more matric units are said to be *connected* if they have the same coefficient (otherwise, a matric unit will be called *isolated*). In the example above, the matric units on the diagonal are connected and the matric units outside the diagonal are not connected. That is, connected matric units yield the linear combinations which generate the subalgebra we consider ($E_{11} + E_{22} + E_{33}$ in the example above).

When dealing with *subrings $\mathcal{S}$ with identity* of full matrix rings $\mathcal{M}_n(R)$ (i.e., $I_n \in \mathcal{S}$), notice the following:

(i) if a matric unit on the diagonal is not connected with other matric units on the diagonal, it must be isolated (i.e., it cannot be connected with any matric unit outside the diagonal);

(ii) if some matric units on the diagonal are connected, matric units outside the diagonal cannot belong to this connection.

The set of matrices $\left\{ \begin{bmatrix} a & a & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} : a \in R \right\} = \{a(I_3 + E_{12}) : a \in R\}$ is a subring, but has no identity (as subalgebra, it has only one generator $E_{11} + E_{22} + E_{33} + E_{12}$).

(iii) a matric unit (or more) on the diagonal, which is connected to some other matric units on the diagonal, may be connected to some matric units outside the diagonal, but under a different connection.

For example, the set $\left\{ \begin{bmatrix} a+b & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} : a, b \in R \right\}$ i.e., linear combinations $aI_4 + b(E_{11} + E_{12})$ with idempotent $E_{11} + E_{12}$.

As observed above, if not isolated, all matric units on the diagonal must be connected with each other, like, $a_1 \sum\limits_{i \in B_1} E_{ii} + ... + a_m \sum\limits_{i \in B_m} E_{ii}$, and these m linear combinations act like δ, i.e., identity or zero, when multiplied with matric units outside the diagonal. That is, for instance $(\sum\limits_{i \in B_1} E_{ii})E_{jk} = \begin{cases} E_{jk} & if \quad j \in B_1 \\ 0 & if \quad j \notin B_1 \end{cases}$, and similarly on the left. This means that when finding conditions which assure an R-module to be a subring with identity, the matric units *connected on the diagonal* are no concern with respect to closure under multiplication (when multiplied by each other, the linear combinations on the diagonal act as idempotent or zero).

Other examples. (a) The set $\left\{ \begin{bmatrix} a_1 & a_1 & ... & a_1 \\ a_2 & a_2 & ... & a_2 \\ \vdots & \vdots & ... & \vdots \\ a_n & a_n & ... & a_n \end{bmatrix} : a_i \in R, 1 \leq i \leq n \right\}$ given in [3], as (general) Armendariz but not (general) reduced (sub)ring, has no identity (it is generated by independent linear combinations: $\{E_{11} + E_{12} + ... + E_{1n}, E_{21} + E_{22} + ... + E_{2n}, ..., E_{n1} + E_{n2} + ... + E_{nn}\}$.

(b) $\left\{ \begin{bmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{bmatrix} | a, a_{ij} \in R \right\}$, $aI_4 + \sum_{i<j} a_{ij}E_{ij}$, or

(c) $\left\{ \begin{bmatrix} a & c & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix} : a, b, c \in R \right\}$, $a(E_{11}+E_{22})+b(E_{33}+E_{44})+c(E_{12}+E_{21})$, or

(d) $\left\{ \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & d & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix} : a, b, c, d \in R \right\}$, $a(E_{11}+E_{22}+E_{33})+bE_{44}+cE_{13}+dE_{23}$.

An *upper triangular Toeplitz* matrix over R is given as

$$\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & ... & a_{n-2} & a_{n-1} & a_n \\ 0 & a_1 & a_2 & ... & a_{n-3} & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & ... & a_{n-4} & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & ... & 0 & a_1 & a_2 \\ 0 & 0 & 0 & ... & 0 & 0 & a_1 \end{bmatrix} : a_i \in R, 1 \leq i \leq n \right\}$$, that is, gen-

erated by the sums $E_{11}+E_{22}+...+E_{nn}, E_{12}+E_{23}+...+E_{n-1,n}, ..., E_{1,n-1}+E_{2n}, E_{1n}$. Here again, this is a subring with identity.

4 Commutative subrings

Further, a binary relation ρ on N will be called *zero square* if $\rho^2=\varnothing$ (in the semigroup with zero $(\mathcal{P}(N\times N),\circ,\varnothing)$). Notice that in this case $\rho\cap\Delta_N=\varnothing$ (i.e., ρ does not contain equal pairs). We call a set of matric units $\{E_{ij},(i,j)\in\rho\}$ *independent*, if ρ is zero square (since $i\neq j$, these are matric units outside the diagonal), and *dependent* otherwise. In this case for any $(i,j),(k,l)\in\rho$, $E_{ij}E_{kl}=0_n$. Notice that *any linear combination of independent matric units is zero square* (more, any two such linear combinations have zero product).

As examples, every matric unit outside the diagonal is zero square. So is $E_{12}+E_{13}$. More general, any outside sum of matric units *on the same row* (or same column) is zero square.

What follows refers to commutative subrings.

Proposition 5 *If a subring $\mathcal{S}$ consists only in symmetric matrices, it is commutative.*

Proof. Since $\mathcal{S}$ is closed under multiplication, for any two (symmetric) matrices $A,B\in\mathcal{S}$, the product is also symmetric. But this happens if and only if $AB=BA$. ■

Further

Proposition 6 *If a subring $\mathcal{S}$ consists only in matrices with scalar diagonal, and outside the diagonal the matric units are independent (connected or not), then $\mathcal{S}$ is commutative.*

Proof. Indeed, such matrices are sums $X+Y$, $X'+Y'$ with scalar X,X', so central, and $Y.Y'=Y'.Y=0$. This way $(X+Y)(X'+Y')=(X'+Y')(X+Y)$. ■

Examples. (1) $\mathcal{F}=\left\{\begin{bmatrix} a & 0 & b & c\\ 0 & a & 0 & d\\ 0 & 0 & a & 0\\ 0 & 0 & 0 & a\end{bmatrix}: a,b,c,d\in R\right\}$, i.e., linear combinations $aI_4+bE_{13}+cE_{14}+dE_{24}$. Notice that here the zero square elements are only the combinations $bE_{13}+cE_{14}+dE_{24}$. The relation $\{(1,3),(1,4),(2,4)\}$ is zero square. So $\mathcal{F}$ is commutative.

(2) $\mathcal{T}_4=\left\{\begin{bmatrix} a & b & x & y\\ 0 & a & b & z\\ 0 & 0 & a & b\\ 0 & 0 & 0 & a\end{bmatrix}: a,b,x,y,z\in R\right\}$. Here the diagonal is scalar, but outside the diagonal we have matric units that are dependent: $\{(1,2),(2,3),(3,4)\}$ is not zero square.

Denoting $N = E_{12} + E_{23} + E_{34} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we obtain $N^2 = E_{13} + E_{24}$, $N^3 = E_{14}$ and $N^4 = 0_4$. This way, an arbitrary matrix in $\mathcal{T}_4$ can be written as $A = aI_4 + bN + xE_{13} + yN^3 + zE_{24}$. For instance, E_{13} is zero square but not central: $NE_{13} = 0_4 \neq E_{14} = E_{13}N$. Here $E_{12}, E_{23}, E_{34} \notin \mathcal{T}_4$ (only the sum $E_{12} + E_{23} + E_{34} \in \mathcal{T}_4$). So $\mathcal{T}_4$ is not commutative. However, it is semicommutative (see [2]).

The following easy properties of matric units are useful when searching for not central nilpotent elements.

Proposition 7 *Let $\mathcal{S}$ be a subring of $\mathcal{M}_n(R)$ for any ring with identity R.*

(i) If i, j, k are distinct among $\{1, 2, ..., n\}$ and $E_{ij}, E_{jk} \in \mathcal{S}$ then E_{ij} is a zero square matrix which is not central.

(ii) If a matric unit $E_{ij} \in \mathcal{S}$ $(i \neq j)$ is central then for all $k, l \in \{1, 2, ..., n\}$, $E_{ki}, E_{jl} \notin \mathcal{S}$, i.e., if there is a central matric unit on the i-th row and j-th column in $\mathcal{S}$, there cannot be other matric units on the j-th row nor on the i-th column in $\mathcal{S}$.

Proof. (i) For $i \neq j$, $E_{ij}^2 = 0_n$ and $E_{ij}E_{jk} = E_{ik} \neq 0_n = E_{jk}E_{ij}$.

(ii) By contradiction, suppose there exists $E_{ki} \in \mathcal{S}$. If $k = j$, then $E_{ji} \in \mathcal{S}$ (the symmetric) and E_{ij} is not central: $E_{ij}E_{ji} = E_{ii} \neq E_{jj} = E_{ji}E_{ij}$. If $k \neq j$, then $E_{ki}E_{ij} = E_{kj} \neq 0 = E_{ij}E_{ki}$ and again E_{ij} is not central. ■

Remark. Any given matric unit E_{rs} $(r \neq s)$ does not commute with E_{sr}: indeed $E_{rs}E_{sr} = E_{rr} \neq E_{ss} = E_{sr}E_{rs}$.

The question, "how can subrings of R^n, which are not subalgebras, be described ?", is not addressed here.

References

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[2] Liu Z. K. *Semicommutative Subrings of Matrix Rings*. Journal of Mathematical Research & Exposition **26**, 2 (2006), 264–268.

[3] Wang W. K. *A Class of Maximal General Armendariz Subrings of Matrix Rings*. Journal of Mathematical Research & Exposition **29**, 1, (2009) 185–190.