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ABSTRACT

We study matrices over general rings which are sums of nilpotent matrices. We show that over commutative rings all matrices with nilpotent trace are sums of three nilpotent matrices. We characterize 2-by-2 matrices with integer entries which are sums of two nilpotents via the solvability of a quadratic Diophantine equation. Some exemples in the case of matrices over noncommutative rings are given.

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1. Introduction

The general question which asks to identify the elements of a ring which can be expressed as a sum of elements which satisfy some special properties (e.g. are units or satisfy some polynomial identities) has a long history, and various particular cases are used in order to study important properties of the ring (e.g. clean rings, nil-clean rings etc.). For instance, for a positive integer k, rings in which each element is the sum of k units were first considered by Henriksen (see [1]), and Vámos (see [2]) called such rings k-*good* rings. In [1] it was proved that for any ring R and any positive integer $n > 1$, the matrix ring $\mathcal{M}_n(R)$ is k-good for $k \leq 3$. Moreover, if R is an elementary division ring then $\mathcal{M}_n(R)$ is 2-good. Similar problems can be stated by replacing units by idempotents or nilpotents. For instance, in [3] it was proved that over infinite-dimensional complex Hilbert spaces, every operator is a sum of at most five idempotents, respectively a sum of at most five square-zero nilpotents. These results were recently generalized in[4], were it is proved that every endomorphism of an infinite-dimensional vector space over a field is the sum of four idempotents and is the sum of four square-zero endomorphisms. For some similar results which involve square-zero nilpotent matrices we refer to [5] and [6].

In this paper we study square matrices over general rings which are sums of nilpotent matrices. It is easy to see that not all square matrices are sums of nilpotents (e.g. matrices over fields have to have zero trace). However, there exist division rings D such that all matrices over D are sums of nilpotents.[7] Therefore, it is interesting to find all matrices which are sums of nilpotents (i.e. to describe the subgroup additively generated by all

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nilpotent matrices), and for a fixed matrix A, to find minimal decompositions for A as sums of nilpotents (i.e. with minimal number of summands). To simplify the writing we shall use the following

Definition: Let k be a positive integer. An element in a ring is called (k-) *nilgood* if it is the sum of (k) nilpotents.

The content of this paper is the following: Section 2 is dedicated to $n \times n$ matrices ($n > 1$) over commutative rings. It is proved that a matrix is a sum of nilpotents if and only if its trace is nilpotent, and all matrices with this property are 3-nilgood. In the case of matrices over fields of zero characteristic every *traceless matrix* (i.e. zero trace) is 2-nilgood, but this may not be true for some positive characteristics (if n is the characteristic of a field then I_n is a traceless matrix which cannot be written as a sum of two nilpotents). Therefore, we characterize 2-nilgood matrices over fields of positive characteristic. Moreover, in the last part of this section we study 2×2 matrices over the ring of integers. Such traceless matrices can be written as a sum of two nilpotent matrices with rational coefficients, but this may fail if we restrict ourselves to nilpotents with integer coefficients. Therefore 2-nilgood matrices over $\mathbf{Z}$ are characterized via the solvability of a quadratic Diophantine equation. This characterization is used to construct some numerical examples which show that the problem of finding all 2-nilgood matrices over $\mathbf{Z}$ or to find all decompositions of such traceless matrices as a sum of two nilpotents is more difficult than it looks at a first sight. In Section 3, some results are given on nilgood matrices over noncommutative rings. In particular it is proved that in the case of 2×2 matrices with entries in the quaternion $\mathbf{R}$-algebra, not all nilgood matrices are 2-nilgood. Moreover, if D is a division ring which contains an element which is a sum of three commutators, but is not a sum of two commutators, then there exists a 4-nilgood matrix which is not 3-nilgood.

Throughout, R denotes a nonzero (associative) ring with identity. The ring of $n \times n$ matrices over R is denoted by $\mathcal{M}_n(R)$. All matrices we consider are square matrices. By $\mathrm{diag}(a_{11}, \ldots, a_{nn})$ we denote an $n \times n$ diagonal matrix (i.e. with only zero entries off the diagonal). Whenever it is more convenient, we use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Matrices over commutative rings

Let R be a commutative ring. If n is a positive integer and $A \in \mathcal{M}_n(R)$, we denote by $\chi_A = \det(XI_n - A)$ the characteristic polynomial associated to A. Recall that in a commutative ring, nilgood elements are nilpotent.

If K is a field, we can deduce from [8] that for every n-tuple $(a_1, \ldots, a_n) \in K^n$ and for every polynomial $f = X^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0 \in K[X]$ such that $r_{n-1} = -a_1 - \cdots - a_n$, there exists a matrix M which has on its diagonal the n-tuple $(a_1, \ldots, a_n)$ and $\chi_M = \det(XI_n - M) = f$. By a similar technique to that used in[9], we can extend this to commutative rings.

Lemma 1: *Let R be a commutative ring. If $a_1, \ldots, a_n \in R$ and $f = X^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0 \in R[X]$ is a monic polynomial such that $r_{n-1} = -a_1 - \cdots - a_n$, then there exists an $(n-1)$-tuple $(b_1, \ldots, b_{n-1}) \in R^{n-1}$such that the characteristic polynomial associated to the matrix*

$$M = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & b_1 \\ 1 & a_2 & 0 & \cdots & 0 & b_2 \\ 0 & 1 & a_3 & \cdots & 0 & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix} \in \mathcal{M}_n(R)$$

is $\chi_M = f$.

Proof: We prove this by induction on n. For $n = 1$ the property is obvious. So we suppose that the property is true for $n - 1$ (with $n \geq 2$) and we prove it for n.

Let $a_1, \ldots, a_n$ be elements from R and let $f = X^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0 \in R[X]$ be a monic polynomial such that $r_{n-1} = -a_1 - \cdots - a_n$. We have to find elements $b_1, \ldots, b_{n-1} \in R$ such that

$$\begin{vmatrix} X - a_1 & 0 & 0 & \cdots & 0 & -b_1 \\ -1 & X - a_2 & 0 & \cdots & 0 & -b_2 \\ 0 & -1 & X - a_3 & \cdots & 0 & -b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & X - a_{n-1} & -b_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & X - a_n \end{vmatrix} = f.$$

This is equivalent to the equality

$$(X - a_1)\begin{vmatrix} X - a_2 & 0 & 0 & \cdots & 0 & -b_1 \\ -1 & X - a_3 & 0 & \cdots & 0 & -b_2 \\ 0 & -1 & X - a_4 & \cdots & 0 & -b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & X - a_{n-1} & -b_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & X - a_n \end{vmatrix} - b_1 = f.$$

Using the remainder theorem for unital commutative rings, there exists a polynomial $g = X^{n-1} + c_{n-2}X^{n-2} + \cdots + c_0 \in R[X]$ of degree $n - 1$ and an element $b \in R$ such that $f = (x - a_1)g + b$. Moreover $c_{n-2} = -a_2 - \cdots - a_n$. By our hypothesis there exist $b_2, \ldots, b_{n-1} \in R$ such that

$$\begin{vmatrix} X - a_2 & 0 & 0 & \cdots & 0 & -b_2 \\ -1 & X - a_3 & 0 & \cdots & 0 & -b_3 \\ 0 & -1 & X - a_4 & \cdots & 0 & -b_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & X - a_{n-1} & -b_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & X - a_n \end{vmatrix} = g,$$

and if we choose $b_1 = -b$, the proof is complete. □

Having this, we can state and prove the main result of this note.

Theorem 2: *The following conditions are equivalent for a matrix A over a commutative ring:*

(a) *A is 3-nilgood;*
(b) *A is nilgood;*
(c) *The trace of A is nilpotent.*

Proof: (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) Over commutative rings, Almkvist proved in [10] that *for any matrix* $A \in \mathcal{M}_n(R)$, $A^{m+1} = 0_n$ *implies* $\mathrm{Tr}(A)^{mn+1} = 0$. Therefore, traces of nilpotent matrices are nilpotent and so are traces of nilgood matrices.

(c) $\Rightarrow$ (a) Let t be the trace of A. By the previous lemma, there exists a matrix $M \in \mathcal{M}_n(R)$ with the same diagonal as A such that $\chi_M = X^n - tX^{n-1}$. Using the Cayley-Hamilton Theorem (see[11], for matrices over commutative rings), it follows that M is nilpotent. Moreover $A - M$ has only 0 on its diagonal, hence it splits as a sum of its upper-triangular and lower-triangular components. Therefore, $A - N$ is a sum of two nilpotent matrices. Hence A is a sum of three nilpotent matrices. □

As an example in the proof of the previous theorem, consider a diagonal diag$(a; b; c)$, whose trace is nilpotent, say $a + b + c = t \in N(R)$, and take

$$N = \begin{bmatrix} a & 0 & a^2(b+c) \\ 1 & b & ab+ac+bc \\ 0 & 1 & c \end{bmatrix}$$

whose characteristic polynomial is $X^3 - (a+b+c)X^2$. Thus N is nilpotent, and all entries on the diagonal of $A - N$ are 0, so $A - N$ is a sum of two nilpotent matrices.

Note that in general this decomposition is not unique. Using a different completion result, proved in[12], we can give an explicit nilpotent completion, namely

$$N' = \begin{bmatrix} a & ab-1 & a+b+a(1-(a+b)^2) \\ 1 & b & 1-(a+b)^2 \\ 0 & 1 & t-a-b \end{bmatrix}.$$

For a direct computation, first replace N' with a conjugate $N_t = UN'U^{-1}$ with $U = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, that is, $N_t = \begin{bmatrix} 0 & -1 & s \\ 1 & s & 1-s^2 \\ 0 & 1 & t-s \end{bmatrix}$ where $s = a + b$. Then it suffices to observe that $N_t^3 = t \begin{bmatrix} 0 & s & * \\ 0 & 1-s^2 & * \\ 1 & t-s & * \end{bmatrix}$.

It is worthwhile mentioning that the 3-nilgood decomposition given in the previous theorem is not always the best (minimal number of nilpotents).

Proposition 3: *Let F be a field and let A be any traceless $n \times n$ matrix over F.*

(i) *If* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F)$ *does not divide n, then A is 2-nilgood.*
(ii) *If* $\mathrm{char}(F)$ *divides n, then all traceless matrices are 3-nilgood, and a matrix A is not 2-nilgood iff $A = \lambda I_n$ for $\lambda \neq 0$.*

Proof:

(i) In,[13] it is proved (see the comment after Theorem 2) that each non-scalar $n \times n$ matrix, over any field, is similar to a matrix with prescribed diagonal entries.

Therefore, if a non-zero matrix A is traceless and gcd $(\mathrm{char}(F), n) = 1$, then it is a non-scalar matrix, and so it is similar to an (also traceless) matrix with zero diagonal entries. Hence it is 2-nilgood.

(ii) The first statement is just a special case of Theorem 2.

For the second statement, we use again [13] to obtain the direct implication. Conversely, if $A = \lambda I_n$ for $\lambda \neq 0$, then for every nilpotent matrix N it is not hard to see that $A-N$ is invertible, hence A is not 2-nilgood. □

In the sequel we investigate the case of 2×2 matrices over $\mathbf{Z}$. A matrix is nilgood iff it is traceless, and (by Theorem 2) every traceless matrix is a sum of three nilpotent matrices. However, despite the fact that the characteristic of $\mathbf{Z}$ is 0, there are traceless 2×2 matrices over $\mathbf{Z}$ which are not sums of two nilpotents.

Example:

(1) If a is odd and b, c are even then the matrix $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is not a sum of two nilpotent matrices over $\mathbf{Z}$, since if we consider these matrices over $\mathbf{Z}/2\mathbf{Z}$ then A represents the identity.

(2) On the other side, there exists A of the above form such that $2A$ is 2-nilgood. A simple example is:

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Thus a natural question is to obtain necessary and sufficient conditions for a traceless matrix $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with integer entries to be 2-nilgood over $\mathbf{Z}$. Moreover, observe that if a, b, and c do not satisfy the conditions in the first example above, then for every prime p, the matrix A represents a matrix over $\mathbf{Z}/p\mathbf{Z}$ which is 2-nilgood, but we will show that this is not sufficient in order to conclude that A is 2-nilgood over $\mathbf{Z}$.

Now, suppose that $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is any traceless matrix over $\mathbf{Z}$. Since nilpotents in $\mathcal{M}_2(\mathbf{Z})$ have the form $N = \begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ with $s^2 + xy = 0$, A is 2-nilgood iff there exist $x, y, s \in \mathbf{Z}$ which satisfy the system

$$\begin{cases} s^2 + xy = 0 \\ 2as + by + cx = -\delta, \end{cases}$$

where $\delta = -a^2 - bc$, i.e. $A-N$ is nilpotent.

Using this, we obtain the following characterization for 2-nilgood matrices over $\mathbf{Z}$:

Theorem 4: *A traceless matrix* $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathcal{M}_2(\mathbf{Z})$, *with* $a \neq 0$, *is 2-nilgood iff the Diophantine equation*

$$c^2X^2 - (4\delta + 2bc)XY + b^2Y^2 + 2c\delta X + 2b\delta Y + \delta^2 = 0 \qquad (*)$$

has solutions in $\mathbf{Z}$.

In this case, for every solution (x, y) *of* $(*)$, *if* $s = \frac{-cx-by-\delta}{2a}$ *then the matrices* $N = \begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ *and* $A - N$ *are nilpotent.*

Proof: Suppose A is 2-nilgood. Then there exist $x, y, s \in \mathbf{Z}$ which verify (S). Since $a \neq 0$ we can write the first equality as $(2as)^2 + 4a^2xy = 0$, further eliminate $2as$ and obtain the Diophantine equation $(*)$.

Conversely, suppose x and y are solutions for the equation $(*)$. If we consider the corresponding equality, and we reduce it modulo $4a^2$ we obtain

$$c^2x^2 + b^2y^2 + 2bcxy + 2c\delta X + 2b\delta Y + \delta^2 \equiv 0 \mod 4a^2,$$

hence $2a \mid cx + by + \delta$.

It follows that $s = \frac{-cx-by-\delta}{2a} \in \mathbf{Z}$, and x, y and s verify the equalities in (S). Hence A is 2-nilgood. □

Remark: The previous theorem can be extended to integral domains whose characteristic is not 2.

Using Proposition 4 together with a computer algorithm for solving quadratic Diophantine equations, we can decide if a matrix is 2-nilgood, and obtain the corresponding decompositions. We list here some simple examples, using [14]:

Example:

(1) Take $a = 1$, $b = c = 3$. Then the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$ is 2-nilgood iff the (symmetric) equation

$$9X^2 + 22XY + 9Y^2 - 60X - 60Y + 100 = 0$$

has solutions.

A family of solutions (x_n, y_n), $n \in \mathbf{Z}$, of this equation is given by the recursive formulas

$$\begin{cases} x_0 = 5,\ y_0 = -5 \\ x_{n+1} = -14x_n - 27y_n + 63 \\ y_{n+1} = 27x_n + 52y_n - 117 \end{cases}$$

In this case we obtain $s_0 = 5$ and in general

$$s_{n+1} = \frac{-3x_{n+1} - 3y_{n+1} + 10}{2} = (-39x_n - 75y_n + 172)/2.$$

We obtain the decomposition

$$\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix} + \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}.$$

(2) In the case $a = 2$, $b = c = 3$ we obtain the (symmetric) equation

$$9X^2 + 31XY + 9Y^2 - 78X - 78Y + 169 = 0$$

which has solutions.

The general solution (x_n, y_n), $n \in \mathbf{Z}$, of this equation is given by the recursive formulas

$$\begin{cases} x_0 = -49,\ y_0 = 16 \\ x_{n+1} = -116x_n - 405y_n + 783 \\ y_{n+1} = 405x_n + 1414y_n - 2727 \end{cases}$$

In the case $n = 0$ we obtain $s_0 = 28$, so

$$\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 28 & -49 \\ 16 & -28 \end{bmatrix} + \begin{bmatrix} -26 & 52 \\ -13 & 26 \end{bmatrix}.$$

(3) In order to find a 2-nilgood decomposition for the matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix}$ we have to solve the Diophantine equation

$$25x^2 + 76xy + 16y^2 - 290x - 232y + 841 = 0.$$

This equation has four principal solutions:

$$(x_0, y_0) \in \{(29, -116), (-725, 261), (528529, -2322576), (-14250625, 5067001)\}.$$

The general solution is given by the formula

$$\begin{cases} x_{n+1} = Px_n + Qy_n + K \\ y_{n+1} = Rx_n + Sy_n + L, \end{cases}$$

where

$$P = -33826519, \quad Q = -95135040, \quad K = 305490640,$$
$$R = 148648500, \quad S = 418064921 \quad L = -1342459300$$

or

$$P = -418064921, \quad Q = -95135040, \quad K = 1073967444$$
$$R = 148648500, \quad S = 33826519, \quad L = -381863295.$$

In particular, we obtain the following "big" decomposition:

$$\begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1107948 & 528529 \\ -2322576 & -1107948 \end{bmatrix} + \begin{bmatrix} -1107945 & -528525 \\ 2322581 & 1107945 \end{bmatrix}.$$

Note that using the general methods presented in[14], we can reduce the equation $(*)$ to the generalized Pell equation

$$U^2 + 4\delta a^2 V^2 - 4c^2\delta a^2 = 0,$$

where $U = 2c^2x - (4\delta + 2bc)y + 2c\delta$, and $V = 2y - c$. It is well known (see [15]) that though this equation has a trivial solution ($U = 0$, $V = \pm c$), it is hard to decide if there exist solutions U and V such that the corresponding solutions x and y are integers. For instance, the above mentioned trivial solution is suitable only if $c \mid a^2$. Moreover, there are matrices A which are not the identity modulo 2 and are not 2-nilgood.

Example:

(1) For $a = 1, b = 2, c = -3$, we obtain the matrix $\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$, which is not 2-nilgood since the corresponding equation,

$$9x^2 - 8xy + 4y^2 - 30x + 20y + 25 = 0$$

has no solutions.
In this case the corresponding Pell equation is $U^2 + 20V^2 - 180 = 0$ has the solutions $(U, V) = (0, \pm 3)$, respectively $(U, V) = \{(\epsilon \cdot 10, \eta \cdot 2) \mid \epsilon, \eta \in \{1, -1\}\}$. In the first case we obtain $y = 0$ and $18x - 30 = 0$ or $y = -3$ and $18x - 6 = 0$, hence $x \notin \mathbf{Z}$. In the second case, since $V = \pm 2$, $2y$ must be odd, hence $y \notin \mathbf{Z}$.

(2) However, even for the elliptic case, it is possible to have solutions. For instance, if $a = 1, b = 3, c = -3$, we have the equation $9x^2 - 14xy + 9y^2 - 48x + 48y + 64 = 0$, which has exactly two solutions $(2, -2)$, and $(1, -1)$. Then there exists a unique decomposition for A:

$$\begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

In some cases it is easy to obtain a decomposition: if $a = b = c$, it is easy to check $\begin{bmatrix} a & a \\ a & -a \end{bmatrix} = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2a & 0 \end{bmatrix}$: such matrices are 2-nilgood.

The following result characterizes 2-nilgood 2×2 traceless upper triangular matrices over $\mathbf{Z}$. The proof given here is independent from the previous theorem.

Proposition 5: *A triangular matrix* $T(a, b) = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ *over* $\mathbf{Z}$ *is not 2-nilgood iff a is odd and b is even.*

Proof: If a is odd and b is even then passing $T(a, b)$ to $\mathbf{Z}/2\mathbf{Z}$ we obtain I_2, which is not 2-nilgood.

Conversely, suppose that a is even or b is odd and let $d = \gcd(a, b)$. Denote $a = a'd$ and $b = b'd$. Then three cases are possible:

I. Suppose $b = 0$. Then

$$T(a, 0) = \begin{bmatrix} a/2 & a/2 \\ -a/2 & -a/2 \end{bmatrix} + \begin{bmatrix} a/2 & -a/2 \\ a/2 & -a/2 \end{bmatrix}.$$

II. Suppose b is odd. Then $d + b'$ is even and

$$T(a, b) = \begin{bmatrix} \frac{1}{2}a'(d + b') & [\frac{1}{2}(d + b')]^2 \\ -a'^2 & -\frac{1}{2}a'(d + b') \end{bmatrix} + \begin{bmatrix} \frac{1}{2}a'(d - b') & -[\frac{1}{2}(d - b')]^2 \\ a'^2 & -\frac{1}{2}a'(d - b') \end{bmatrix}.$$

III. Suppose $b \neq 0$ is even. Then a is even, hence d is also even. If b' is even, then we can take the same matrix as in (II).

If b' is odd then $T(a,b) = dT(a',b')$, and the matrix $T(a',b')$ is a sum of two nilpotent matrices by (II). □

3. Matrices over noncommutative rings

In this section, we investigate matrices over noncommutative rings. Recall that in the case of division rings it was proved in [7, Proposition 2] that a matrix is a sum of nilpotents iff its trace is a sum of commutators, i.e. a sum of elements of the form $[a,b] = ab - ba$. In this context, it is natural to mention a somewhat similar study, done by K. Mesyan in [16] . In his paper, a (noncommutative) ring is called a *commutator* ring if all its elements are sums of commutators. Among other things, for an arbitrary ring R, it is proved that a matrix $A \in \mathcal{M}_n(R)$ is a sum of commutators iff the trace of A is a sum of commutators.

One of the ingredients used in [7, Proposition 2] holds for any ring.

Proposition 6: *Let R be any ring, $n > 1$ and $A \in \mathcal{M}_n(R)$. The matrix A is nilgood iff* $\mathrm{diag}(0,\dots,0,\mathrm{Tr}(A))$ *is nilgood.*

Moreover, if $\mathrm{diag}(0,\dots,0,\mathrm{Tr}(A))$ *is k-nilgood then A is $(n+1+k)$-nilgood.*

Proof: Denote $A = \left[a_{ij}\right]$, $1 \le i,j \le n$. There is a decomposition

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 & a_{11} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -a_{11} & 0 & \dots & 0 & -a_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & a_{22} & \dots & 0 & a_{22} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & -a_{22} & \dots & 0 & -a_{22} \end{bmatrix} + \dots$$

$$+ \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n-1} \\ 0 & 0 & \dots & -a_{n-1,n-1} & -a_{n-1,n-1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & a_{12} & \dots & a_{1,n-1} & a_{1n} - a_{11} \\ 0 & 0 & \dots & a_{2,n-1} & a_{2n} - a_{22} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} - a_{n-1,n-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_{21} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & 0 & 0 \\ a_{n1} + a_{11} & a_{n2} + a_{22} & \dots & a_{n,n-1} + a_{n-1,n-1} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathrm{Tr}(A) \end{bmatrix}.$$

Now the conclusion is obvious. □

Corollary 7: *If R is any ring and $n > 1$, then*

(a) *every traceless matrix $A \in \mathcal{M}_n(R)$ is $(n+1)$ -nilgood;*
(b) *if $\mathrm{Tr}(A)$ is k-nilgood then A is $(n+k+1)$ -nilgood.*
(c) *if $\mathrm{Tr}(A)$ is a sum of k commutators then A is $(n+2k+1)$-nilgood.*

Proof: The statements (a) and (b) are obvious.

For (c), recall from the proof of the mentioned result from[7, Proposition 1], that for every commutator x, the matrix $\mathrm{diag}(0,\dots,0,x)$ is a sum of four nilpotent matrices, and two of them are strictly triangular matrices. □

We were not able to extend Harris's characterization from division rings to general rings. A first obstruction is the fact that, while in the commutative case the matrices with nilpotent trace are nilgood (see Theorem 2), in the general case, a matrix A may not be nilgood when $\mathrm{Tr}(A)^k$ is a sum of commutators for some positive integer k.

Example: Let **H** be the quaternion division ring. It is not hard to see that a quaternion $q \neq 0$ is a sum of commutators iff it is purely imaginary. Therefore, a matrix over **H** is nilgood iff its trace is 0 or purely imaginary. On the other side, $(1+i)^2 = 2i$, but clearly a matrix whose trace is $1+i$ is not nilgood.

Often a better (fewer summands) decomposition is available.

Proposition 8: *Suppose $k > 1$. A matrix is k-nilgood whenever its diagonal entries are k -nilgood.*

Proof: For an $n \times n$ matrix $A = [a_{ij}]$, decompose the entries on the diagonal into k nilpotents. Then decompose the matrix into k diagonal matrices with only nilpotent entries on the diagonal and add to the first and second such matrices the strictly upper triangular respectively strictly lower triangular parts of A, i.e. if $a_{ii} = \sum_{m=1}^{k} t_{im}$ for every $1 \le i \le n$,

$$A = \begin{bmatrix} t_{11} & a_{12} & \dots & a_{1,n-1} & a_{1n} \\ 0 & t_{21} & \dots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & t_{n-1,1} & a_{n-1,n} \\ 0 & 0 & \dots & 0 & t_{n1} \end{bmatrix} + \begin{bmatrix} t_{12} & 0 & \dots & 0 \\ a_{21} & t_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & 0 \\ a_{n1} & a_{n2} & \dots & t_{n2} \end{bmatrix} + \begin{bmatrix} t_{13} & 0 & \dots & 0 & 0 \\ 0 & t_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & t_{n-1,3} & 0 \\ 0 & 0 & \dots & 0 & t_{n3} \end{bmatrix} + \dots + \begin{bmatrix} t_{1k} & 0 & \dots & 0 & 0 \\ 0 & t_{2k} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & t_{n-1,k} & 0 \\ 0 & 0 & \dots & 0 & t_{nk} \end{bmatrix}.$$

□

Next, consider 2×2 matrices over a division ring D. First recall from [17] that if D is a division ring, then nilpotent 2×2 matrices over D can be of the form

$$N(z) = \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix} \text{ or } N(y,z) = \begin{bmatrix} -yz & y \\ -zyz & zy \end{bmatrix},$$

with $y, z \in D$.

Assume $a \neq 0$ is an element in D and let $A = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(D)$. Suppose that A has the form

$A = N(y_1, z_1) + \cdots + N(y_k, z_k) + N(z)$, where $y_1z_1, \ldots, y_kz_k$ are non-zero elements from D, and $z \in D$.

Then we obtain the equalities

$$\begin{cases} y_1z_1 + y_2z_2 + \cdots + y_kz_k = a \\ z_1y_1 + z_2y_2 + \cdots + z_ky_k = 0 \\ y_1 + y_2 + \cdots + y_k = 0 \\ z_1y_1z_1 + z_2y_2z_2 + \cdots + z_ky_kz_k - z = 0. \end{cases}$$

Using the first three equalities we deduce

$$\begin{cases} y_1(z_1 - z_k) + \cdots + y_{k-1}(z_{k-1} - z_k) = a \\ (z_1 - z_k)y_1 + \cdots + (z_{k-1} - z_k)y_{k-1} = 0. \end{cases}$$

In particular a is a sum of $k - 1$ commutators, and we are ready to prove the following special case.

Proposition 9: *Assume $a \neq 0$ is an element of D which is a sum of commutators, but not a sum of two commutators. Then $A = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(D)$ is not 3-nilgood.*

Proof: Suppose by contradiction that A is a sum of three nilpotents. Therefore, we have a decomposition as in the above discussion with $k \leq 3$. In particular a can be written as a sum of at most two commutators, a contradiction. □

As the previous proposition shows, in order to prove that there are nilgood matrices over division rings which are not 3-nilgood, it would suffice to find an example of a division ring D and an element $a \in D$ which is a sum of three commutators but is not a sum of two commutators. We are not aware of any example.

In closing, recall that all nilgood matrices over *fields* of characteristic 0 are 2-nilgood. In the case of *division rings* of 0 characteristic, this is no longer true.

Example. The matrix $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbf{H})$ is nilgood since its trace is a commutator but is not 2-nilgood.

Indeed, suppose A is 2-nilgood. Then $A = N(y_1, z_1) + N(z)$ or $A = N(y_1, z_1) + N(y_2, z_2)$. In the first case we obtain $y_1z_1 = -i$ and $z_1y_1 = 0$, a contradiction. If we suppose that $A = N(y_1, z_1) + N(y_2, z_2)$ we obtain the equalities $y_1(z_1 - z_2) = -i$ and $(z_1 - z_2)y_1 = 0$, a contradiction.

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