

# SPLIT-EXTENSIONS OF EXCHANGE RINGS: A DIRECT PROOF

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ABSTRACT. A direct ring-theoretic proof for "if  $R$  is a ring,  $e$  is an idempotent and  $eRe$  and  $(1-e)R(1-e)$  are both exchange rings then  $R$  is also an exchange ring" (left open for the last 31 years) is given.

The exchange property was first (introduced and) studied for modules by **Crawley P.** and **Jónsson B.** in [1]: for a given cardinal  $m$ , a right  $R$ -module  $M_R$  has the  $m$ -exchange property if whenever  $A = M \oplus N = \bigoplus_{i \in I} A_i$  with  $|I| \leq m$ , there are submodules  $A'_i \leq A_i$  with  $A = M \oplus (\bigoplus_{i \in I} A'_i)$ . If  $M$  has  $m$ -exchange for all cardinals  $m$ , we say  $M$  has the *full exchange* property. If the same holds just for finite cardinals, we say  $M$  has the *finite exchange* property. Further on, **Warfield R. B., Jr.** in [5], considered this property for rings (with identity): a ring  $R$  such that  $R_R$  has finite exchange is called (right) *exchange ring*, and this turned out to be a left-right symmetric condition.

In his 1977 seminal paper "Lifting idempotents and exchange rings", among other things (starting point of clean rings theory, characterization of exchange rings by lifting idempotents modulo one-sided ideals, etc), **Nicholson** calls a ring (right) *suitable* if for each equation  $a + b = 1$  there are (orthogonal) idempotents  $e \in aR$  and  $f \in bR$  such that  $e + f = 1$ . Suitable rings and exchange rings coincide.

In the sequel, by exchange ring we mean right exchange (suitable) ring and assume/check this last (equivalent) condition.

From this paper we quote: "It would be of interest to see a direct ring-theoretic proof of the fact that *if  $R$  is a ring,  $e \in R$  is an idempotent and  $eRe$  and  $(1 - e)R(1 - e)$  are both exchange rings then  $R$  is also an exchange ring*". No such direct proof seems to have appeared the last 31 years. This paper closes this gap.

## 1. THE CLAIM

For convenience write  $\bar{r} = 1 - r$  for each  $r \in R$ . For an idempotent  $e \in R$ , we use the Pierce decomposition of the ring  $R$ :

$$R = \begin{bmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{bmatrix}.$$

Let  $A = \begin{bmatrix} a & x \\ y & \alpha \end{bmatrix} \in R$ . Since  $a \in eRe$  (ring with identity  $e$ ) is exchange, there are  $b, c \in eRe$  such that  $bab = b$  and  $(e - a)c = e - ab$ . The element  $c$  can be chosen

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(see Section 2 (b)), such that  $c = c(e - a)c$ . Consider  $\alpha_1 = \alpha - y(b - c)x \in \bar{e}R\bar{e}$ . By exchange property, there are  $\beta, \gamma \in \bar{e}R\bar{e}$  such that  $\beta\alpha_1\beta = \beta$  and  $(\bar{e} - \alpha_1)\gamma = \bar{e} - \alpha_1\beta$ . Once again (by Section 2 (b)),  $\gamma = \gamma(\bar{e} - \alpha_1)\gamma$  can be assumed.

In order to simplify the writing, denote  $s = x(\beta - \gamma)$  and  $t = y(b - c)$  and consider the invertible matrices  $U = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$  and  $V = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ .

Taking

$$C = \begin{bmatrix} c + cx(\beta - \gamma)y(b - c) & -cx(\beta - \gamma) \\ -\gamma y(b - c) & \gamma \end{bmatrix} = \begin{bmatrix} c + cst & -cs \\ -\gamma t & \gamma \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & \gamma \end{bmatrix} UV$$

and

$$B = \begin{bmatrix} b + bx(\beta - \gamma)y(b - c) & -bx(\beta - \gamma) \\ -\beta y(b - c) & \beta \end{bmatrix} \begin{bmatrix} b + bst & -bs \\ -\beta t & \beta \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & \beta \end{bmatrix} UV,$$

one verifies the equalities

$$BAB = B \text{ and } I_2 - AB = (I_2 - A)C \text{ where } I_2 = \begin{bmatrix} e & 0 \\ 0 & \bar{e} \end{bmatrix}.$$

Therefore,  $E = AB \in AR$  is an idempotent and  $I_2 - E \in (I_2 - A)R$ , as desired.

## 2. THE VERIFICATION

Since (despite the easy claim above), the verification of these two matrix equalities is far from obvious, in this section we supply the necessary details.

We first recall some well-known relations which hold in any exchange ring.

a) If  $e \in aR$  is an idempotent, an element  $b \in R$  can be chosen such that  $e = ab$  and  $bab = b$ .

b) If  $1 - e \in (1 - a)R$  is idempotent, an element  $c \in R$  can be chosen such that  $1 - e = (1 - a)c$  and  $c(1 - a)c = c$ .

c)  $cab = 0$  [using  $c(1 - a)c = c$  and left multiplication by  $c$  in  $1 - ab = (1 - a)c$ ].

Hence, replacing  $R$  by the corner ring  $eRe$ , these relations hold for the elements  $a, b$ , and  $c$  in the above claim (written  $e - ab = (e - a)c$ , from now on denoted (a'))

Similar properties hold for the Greek letters in the above claim (now replacing  $R$  by  $\bar{e}R\bar{e}$ ).

Notice the following relations:

d)  $cab = 0$  implies  $tab = yb$ .

e) Using  $\alpha_1 = \alpha - tx$  in  $(\bar{e} - \alpha_1)\gamma = \bar{e} - \alpha_1\beta$ , gives  $(\bar{e} - \alpha)\gamma = \bar{e} - \alpha\beta + ts$ .

f) Using it in  $\beta\alpha_1\beta = \beta$  gives  $\beta\alpha\beta = \beta + \beta tx\beta$  and,

g)  $\gamma\alpha_1\beta = 0$  and finally  $\gamma\alpha\beta = \gamma tx\beta$ .

**The equality  $BAB = B$ .**

$$\text{Equivalently, we have to check } \begin{bmatrix} b & 0 \\ 0 & \beta \end{bmatrix} UV A \begin{bmatrix} b & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & \beta \end{bmatrix}.$$

11-entry: follows from  $bab = b$  (a) and  $tab = yb$  (d).

12-entry:  $bx\beta + bstx\beta - bs\alpha\beta = 0$ , follows replacing  $s = x(\beta - \gamma)$  from  $\gamma\alpha\beta = \gamma tx\beta$  (g) and  $\beta\alpha\beta = \beta + \beta tx\beta$  (f).

21-entry: follows from  $tab = yb$  (d).

22-entry:  $\beta\alpha\beta = \beta + \beta tx\beta$  (f).

**The equality**  $I_2 - AB = (I_2 - A)C$ .

Equivalently, we have to check  $V^{-1}U^{-1} - A \begin{bmatrix} b & 0 \\ 0 & \beta \end{bmatrix} = (I_2 - A) \begin{bmatrix} c & 0 \\ 0 & \gamma \end{bmatrix}$  which amounts to  $(e - a)c = e - ab$  (a'), our notations  $s, t$  and  $(\bar{e} - \alpha)\gamma = \bar{e} - \alpha\beta + ts$  (e).

### 3. FINAL REMARKS

1) If  $eRe$  and  $\bar{e}R\bar{e}$  are both clean, then the elements considered above have (understandably, since clean rings are exchange) stronger properties:  $b - c = u_1$  is a unit in  $eRe$  and so is  $\beta - \gamma = v_1$  (in  $\bar{e}R\bar{e}$ ). If  $u$  is the inverse of  $u_1$ ,  $v$  is the inverse of  $v_1$  and  $\varepsilon = -cu$ ,  $g = -\gamma v$  then  $A = \begin{bmatrix} a & x \\ y & \alpha \end{bmatrix}$  decomposes into a sum of an idempotent and a unit:  $E + U = \begin{bmatrix} \varepsilon & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} u & x \\ y & v + y(b - c)x \end{bmatrix}$ . As such, it is also clean.

In this way we recover the proof of Lemma 1 in [2].

2) Actually, in the above mentioned paper, Nicholson also asks for a direct ring-theoretic proof for "the (finite) exchange property is a left-right symmetric condition". It took 20 years until Nicholson himself gave this direct proof (see [4]).

### REFERENCES

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