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Solitary subgroups of Abelian groups

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Abstract

Since solitary subgroups of (infinite) Abelian groups are precisely the strictly invariant subgroups which are co-Hopfian (as groups), and strictly invariant subgroups turn out to be strongly invariant for large classes of Abelian groups we determine the solitary subgroups for these classes of groups.

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1. Introduction

Kaplan and Levy [13] gave the following definition: a subgroup K is *solitary* in a group G if G contains no another subgroup isomorphic to K (i.e., for any subgroup H of G, $K \cong H$ implies $K = H$). Solitary subgroups and related subjects were investigated in [1,8,13] and [16], but only for finite groups. While it was immediately noticed that every solitary subgroup must be characteristic (and so normal), dealing *only* with finite groups, the following simple remark was not in use: *every solitary subgroup is co-Hopfian* (as a group — see Theorem below).

Here, a group G is said to be *co-Hopfian* (S-group in [2]) if it is not isomorphic to any of its proper subgroups (or equivalently, if every injective endomorphism $G \longrightarrow G$ is an

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automorphism). Since every finite group is co-Hopfian, this notion is clearly irrelevant for finite groups.

For a subgroup N of a group G the inclusion $f(N) \leq N$ makes sense for f in various sets of group homomorphisms. Some of them (are well-known and other) turn out to be useful for a global characterization of solitary subgroups. These are: $\mathrm{Inj}(N, G) \subseteq \mathrm{Hom}(N, G)$, that is, the injective group homomorphisms $N \longrightarrow G$ and all the group homomorphisms $N \longrightarrow G$, respectively, and, $\mathrm{Aut}(G) \subseteq \mathrm{Inj}(G, G) \subseteq \mathrm{End}(G)$, with a similar notation for the injective endomorphisms of G. The corresponding classes of subgroups are called: strictly invariant, strongly invariant, and, characteristic, injective invariant (S-characteristic in [2], left invariant in [11]) and fully invariant, respectively. More precisely

Definition. A subgroup N is *strongly invariant* (see [6]) in a group G if $f(N) \leq N$ for every group homomorphism $f : N \longrightarrow G$. Equivalently, for any subgroup H of G, $H \leq N$ whenever a subgroup epimorphism $N \longrightarrow H$ exists; a subgroup N is called *strictly invariant* in a group G if $f(N) \leq N$ for every injective group homomorphism $f : N \longrightarrow G$. Equivalently, for any subgroup H of G, $H \leq N$ whenever a subgroup isomorphism $N \longrightarrow H$ exists.

With the above definitions we have the following chart

$$\begin{array}{ccccccc} & & \text{strongly-invariant} & \Rightarrow & \text{fully-invariant} & & \\ & & \Downarrow & & \Downarrow & & \\ \text{solitary} & \Rightarrow & \text{strictly-invariant} & \Rightarrow & \text{injective-invariant} & \Rightarrow & \text{characteristic} \end{array}$$

From definitions follows the characterization

Theorem 0. A subgroup K is solitary in a group G if and only if it is strictly invariant and co-Hopfian (as a group).

Since every group is strictly invariant in itself we obtain at once something different from the finite groups case (where G is always solitary in G).

Corollary 1. *G is solitary in G iff G is co-Hopfian.*

However, since finite groups are co-Hopfian

Corollary 2. *A finite subgroup is solitary in a group iff it is strictly invariant.*

Note that according to Theorem 14, (a), we can replace strictly invariant in the previous corollary, by strongly invariant.

In this paper we determine the solitary subgroups of mostly all classes of (infinite) Abelian groups. While the astute reader will easily observe that some of our results hold also for not necessarily commutative groups, in order to simplify the writing, we state and prove everything in the commutative setting. Some of the results hold not only for $\mathbb{Z}$ -modules (i.e., Abelian groups) but also for general modules over rings (with identity).

The characterization above explains why results on co-Hopfian (Abelian) groups and strictly (strongly) invariant subgroups are needed in order to determine the solitary subgroups of (Abelian) groups. Such results are surveyed in Sections 2 and 3, respectively.

After mentioning some general properties of solitary subgroups, in Section 4, we present the proofs of our main results on solitary subgroups of Abelian groups, which we summarize below. For a subgroup H of a group G, remind that the G-socle of H is $S_G(H) = \sum_{f:G\to H} \operatorname{Im} f$.

The discussion of solitary subgroups of (Abelian) groups can be made into *two directions*: a special class of groups is considered (e.g., divisible or torsion-free or torsion or mixed) and we determine all solitary subgroups, or else, for a (more or less) arbitrary group we determine solitary subgroups which belong to a special class.

In the first direction, our results are gathered in the next

Theorem 3. *(A) Let G be a torsion-free group.*

(1) If G is reduced then 0 is the only solitary subgroup of G.

(2) If G is divisible then G is solitary simple iff G has finite rank.

(3) If G is not reduced then the only solitary subgroup of G is $D(G)$, the divisible part, if it is of finite rank.

(B) (1) Let H be a torsion subgroup of a group G. Then H is solitary in G iff each p-component H_p is solitary in G_p.

(2) Let H be a subgroup of a reduced p-group G. Then H is solitary in G iff $H = G[p^n]$ for some positive integer n and H is finite or else $H = G$ and $\operatorname{card} H = 2^{\aleph_0}$.

(C) The solitary subgroups of a splitting reduced mixed group G are the solitary subgroups of the torsion part if the 2-component of G, $G_2 = \{a \in G : \exists n \geq 0, a^{2^n} = 0\} = 0$.

These can be found (with proofs) as Theorem 17, Proposition 18 , Theorems 20 and 21.

In the second direction, our results are gathered in

Theorem 4. *(D) (i) The only solitary divisible p-subgroup of a group G is the (whole) divisible p-part, i.e., $D(G_p)$, if it is of finite rank.*

(ii) The only solitary divisible torsion-free subgroup of a group is the (whole) divisible torsion-free part, if it is of finite rank.

(E) The divisible part $D(G)$ of a group G is solitary, iff $D_p(G)$ is of finite rank for all primes p and $r_0(D(G))$ (the torsion-free rank of $D(G)$) is finite.

(F) Let H be a subgroup of $G = D(G) \oplus R$ with divisible part $D(G)$ and reduced direct complement R. If $H = D(H) \oplus L$ with L chosen as a subgroup of R then H is solitary in G iff $D(H)$ is solitary in $D(G)$, L is solitary in R and $S_L(D(G)) \leq D(H)$.

(G) The p-components of a torsion group G are solitary subgroups iff G is co-Hopfian.

(H) (i) If $G = T(G) \oplus F$ with reduced torsion part $T(G)$ and torsion-free direct complement F, T is solitary in $T(G)$, $S_K(T(G)) \leq T$ and K is solitary in F then $T \oplus K$ is solitary in G.

(ii) If $G = T(G) \oplus F$ with reduced torsion part $T(G)$ and torsion-free direct complement F, T is solitary in $T(G)$ and K is solitary in F then $T \oplus K$ is solitary in G.

These can be found (with proofs) as Proposition 22, Corollary 23, Propositions 24, 26, 27 and Corollary 28.

The section ends with an example of a solitary genuine mixed subgroup of a co-Hopfian group.

Finally, in the last Section, we determine the solitary simple Abelian groups, that is, the Abelian groups with no proper solitary subgroups, and the Abelian groups all whose subgroups are solitary.

All the groups we consider are Abelian (unless otherwise stated). For notions and results on Abelian groups we refer to the comprehensive treatise of L. Fuchs, [9,10]. As in [9], a group is called *countable* if it is finite or of cardinality $\aleph_0$. A group G is genuine mixed if $0 \neq T(G) \neq G$ with torsion part $T(G)$.

We will use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Co-Hopfian Abelian groups

First recall (mostly Varadarajan [7,17,18]) some well-known results.

Proposition 5. *(i) A direct decomposition $G = \bigoplus_{i\in I} H_i$ with all H_i fully invariant summands is co-Hopfian iff every H_i is co-Hopfian.*

(ii) Let G be a torsion group. Then G is co-Hopfian iff all primary components G_p are co-Hopfian.

(iii) Given any group $0 \neq G$, any infinite direct sum (or product) of copies of G is not co-Hopfian.

(iv) A divisible group D is co-Hopfian iff it is of finite torsion-free rank, and of finite p-rank, for every prime number p, that is, $D \cong \mathbb{Q}^n \oplus (\bigoplus_{p\in\mathbf{P}} \mathbb{Z}(p^\infty)^{n(p)})$ with positive integers $n, n(p)$.

(v) Any direct summand of a co-Hopfian group is co-Hopfian.

(vi) Let $G = H \oplus K$ with fully invariant H. If both H and K are co-Hopfian then so is G.

(vii) nG is co-Hopfian whenever G is co-Hopfian.

(viii) Generally, quotients of co-Hopfian groups may not be co-Hopfian.

Examples. Both $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are co-Hopfian, and $\bigoplus_{p\in\mathbf{P}} \mathbb{Z}(p)$ is co-Hopfian. Since every Artinian module is co-Hopfian (the converse fails: $\mathbf{Q}$), finitely cogenerated groups are co-Hopfian. $\mathbb{Z}$ is not co-Hopfian. Hence, every finitely generated *infinite* Abelian group is not co-Hopfian (more, a finitely generated group is co-Hopfian iff it is finite). For any group G, if a group T is not co-Hopfian, the direct product (or sum) $G \times T$ is not co-Hopfian.

Next we mention the following early result of R. Baer [2] (and some important consequences)

Theorem 6. *The group G is co-Hopfian if there exists a well ordered ascending chain $N(v)$ of* injective invariant *subgroups of G with the following properties.*

(i) $N(0) = 1$ and $N(t) = G$ for some ordinal t.

(ii) $N(v + 1)/N(v)$ is co-Hopfian.

(iii) If v is a limit ordinal, then every element in $N(v)$ is contained in some $N(u)$ for $u < v$.

Corollary 7. *(a) Let $H \rightarrowtail G \twoheadrightarrow K$ be an exact sequence of groups with K and H co-Hopfian. If H is injective invariant then G is co-Hopfian.*

(b) Let $G = D(G) \oplus R$ the direct decomposition of G with divisible part $D(G)$ and a reduced direct complement R. Then G is co-Hopfian iff both $D(G)$ and R are co-Hopfian.

(c) A splitting mixed group $G = T(G) \oplus F$ is co-Hopfian iff both $T(G)$ and F are co-Hopfian.

(d) Let G be a mixed group. If both $T(G)$ and $G/T(G)$ are co-Hopfian, so is G.

Therefore, the usual reductions for the determination of the co-Hopfian Abelian groups work: it suffices to find the reduced p-groups and the reduced nonsplitting groups.

Further, since an (Abelian) group is torsion-free (or divisible) exactly when all multiplications with positive integers are injective (respectively surjective), we can easily dispose of co-Hopfian torsion-free groups

Proposition 8. *Every torsion-free co-Hopfian group is divisible.*

Corollary 9. *(i) The only co-Hopfian torsion-free groups are the finite direct sums $\mathbb{Q} \oplus \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$.*

(ii) The only (genuine) splitting mixed co-Hopfian groups are of the form $G = T(G) \oplus (\mathbb{Q} \oplus \mathbb{Q} \oplus \cdots \oplus \mathbb{Q})$ with only finitely many $\mathbb{Q}$.

Concerning infinite co-Hopfian p-groups we mention (for a proof see [12]):

Theorem 10. *There are no infinite reduced co-Hopfian p-groups G such that* $\mathrm{card} G = \aleph_0$ *or* $\mathrm{card} G > 2^{\aleph_0}$.

Corollary 11. *(a) Any reduced countable co-Hopfian p-group is finite.*

(b) Reduced co-Hopfian p-groups have finite Ulm invariants and so have cardinality at most $2^{\aleph_0}$.

Therefore, since finite groups are co-Hopfian, the determination of the infinite reduced co-Hopfian p-groups, reduces (assuming the continuum hypothesis) to p-groups of power of continuum. Such co-Hopfian p-groups do exist: in the of the torsion part of $\prod_{k=1}^{\infty} \mathbb{Z}(p^k)$ (an example of p-group of the power of continuum, without elements of infinite height, which is not a direct sum of cyclic groups, which comes back to Kurosh [14], see also **17.3**, [9]), Crawley (see [5]) constructed a pure subgroup of the power of continuum, without elements of infinite height which has no proper isomorphic subgroups.

Finally notice that

Proposition 12. *For a (reduced) co-Hopfian group G, $T(G)$ and $G/T(G)$ may not be co-Hopfian.*

Indeed, in [11], some mixed groups are constructed which have the property that these are extensions of a not co-Hopfian fully invariant subgroup by a not co-Hopfian group, and yet are co-Hopfian.

Remarks. (1) Subgroups of co-Hopfian groups may not be co-Hopfian (e.g. $\mathbb{Z}$ as subgroup of $\mathbb{Q}$). As a special case, if G is a co-Hopfian unbounded p-group (for existence see Pierce [15]), then any basic subgroup B of G is an unbounded direct sum of cyclic p-groups, and so is not co-Hopfian.

(2) Any unbounded direct sum of cyclics is not co-Hopfian.

3. Strictly (strongly) invariant subgroups

Recall that a subgroup N was called *strictly invariant* in a group G if $f(N) \leq N$ for every injective group homomorphism $f : N \longrightarrow G$. Equivalently, for any subgroup H of G, $H \leq N$ whenever a subgroup isomorphism $N \longrightarrow H$ exists.

The strictly invariant subgroups form a complete lattice and so arbitrary sums of strictly invariant subgroups are strictly invariant. Moreover, the strictly invariant property for subgroups is not transitive.

Strictly invariant submodules were recently studied in [3] with special emphasis to Abelian case. While an example of strictly invariant submodule which is not strongly invariant was given, the authors were not able to construct such an example for the case of Abelian groups. Therefore the paper was focussed on finding fairly general conditions on the group and/or on the subgroup, which imply that the strictly invariant subgroups are strongly invariant, in order to argue the enunciation of the following

Conjecture 13. *Any strictly invariant subgroup of an Abelian group is strongly invariant.*

Very large classes of Abelian groups are shown to support this conjecture.

We just list these (from [3]) in what follows.

Theorem 14. *Let H be a strictly invariant subgroup of a group G. Then H is strongly invariant in any of the situations listed below.*

(a) H is torsion.

(b) $G = D(G) \oplus R$ with reduced R and $r_p(D(G)) \geq \max\{r_p(R), \aleph_0\}$ for every $p \in \mathbb{P} \cup \{0\}$.

(c) G is torsion-free and H has finite rank.

(d) G is torsion-free and all rank 2 pure subgroups of G are indecomposable.

(e) G is torsion-free separable.

It is also proved that *a subgroup of a completely decomposable group is strictly invariant iff it is a fully invariant direct summand* (so more than strongly invariant).

Therefore, for all groups in the list above, *a subgroup is solitary iff it is strongly invariant and co-Hopfian*, which means that we have to use the results on strongly invariant subgroups obtained in [6] and [4]. We abbreviate "strongly invariant" by "s-i" in the next

Summary. 1. (a) Torsion-free divisible groups are strongly invariant simple.

(b) The only (proper) s-i subgroups of a divisible p-group G are the subgroups $G[p^{n_p}]$ for positive integers n_p.

(c) The s-i subgroups of a divisible group G are G itself and the s-i subgroups of its torsion part $T(G)$, i.e., direct sums of $\bigoplus_p A_p$, where $A_p = G_p$ or $A_p = G[p^{n_p}]$ for positive integers n_p.

2. (i) Let $G = D(G) \oplus R$ be a decomposition of a group G with $D(G)$ its divisible part and R a reduced group. Every s-i subgroup N of G has the form $N = D_1 \oplus R_1$ with D_1 s-i subgroup in $D(G)$, R_1 s-i subgroup in R. Conversely, a direct sum $D_1 \oplus R_1$ with D_1 s-i subgroup in $D(G)$, R_1 s-i subgroup in R is a s-i subgroup in G iff $S_{D_1}(R) \leq R_1$ and $S_{R_1}(D(G)) \leq D_1$.

(ii) A direct sum $D' \oplus R' \neq G$ with D' s-i subgroup in $D(G)$, R' s-i subgroup in R is a s-i subgroup in G iff for every prime number p, if $D' = \bigoplus \mathbb{Z}(p^n)$, then $S_{R'}(\mathbb{Z}(p^\infty)) \leq \mathbb{Z}(p^n)$ and $S_{D'}(R) \leq R'$.

3. (i) Let A be a subgroup in a torsion group G. Then A is s-i in G iff for every prime p, A_p is s-i in G_p.

(ii) The only (proper) s-i subgroups of a reduced p-group are the subgroups $G[p^n]$ for all positive integers n.

4. Let N be a subgroup of a mixed group G. Then $T(N)$ is s-i subgroup of $T(G)$ iff $T(N)$ is s-i subgroup of G.

Hence, the torsion s-i subgroups in a mixed group are only the subgroups $G[n]$ for all positive integers n.

5. If in a (genuine) mixed group a subgroup contains a free direct summand (e.g. it is infinite cyclic), it is not s-i.

4. Standard properties and proofs

We first mention some *standard properties* for solitary subgroups

Proposition 15. *(a) Solitary subgroup is not a* transitive *property.*

(b) Solitary subgroups have the intermediate subgroup property*, i.e., if $H \leq K \leq G$ and H is a solitary subgroup in G then H is also solitary in K.*

(c) The solitary subgroups of a group G form a sublattice of the subgroup lattice $L(G)$, which is generally not complete.

(d) Solitary subgroups are quotient transitive*, i.e., if $K \leq H \leq G$, K is solitary in G and H/K is solitary in G/K, then H is solitary in G. In particular, if K is maximal solitary in G, then G/K is solitary simple.*

Proof. (a) Consider $G = H \oplus L = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2)$ with $K = \mathbb{Z}(2) < H$. Then $K = S(H)$, the socle, is strongly and so strictly invariant in H. It is not strictly invariant in G, since the composition of the isomorphism $K \cong L$ with the injection $i_L : L \longrightarrow G$ does not map K into K. Finally, H is a fully invariant direct summand – as divisible part of G – and so strongly and strictly invariant in G.

Generalization. Let D be divisible and K a reduced strictly invariant subgroup of D. Take $G = D \oplus K$.

(b) Obvious.

(c) One uses "for S_i, $i \in I = \{1, \ldots, n\}$ and T subgroups in G, let $T \cong \sum_{i=1}^{n} S_i$. Then T contains subgroups T_i ($i \in I$) which are copies of S_i (i.e., $T_i \cong S_i$), respectively, and $T = \sum_{i=1}^{n} T_i$", in order to adapt the proof given in [13] (**Th. 25**). Note that $\inf\{S_i\}_{i \in I}$ is not necessarily the intersection $\bigcap S_i$.

Since infinite sums of co-Hopfian groups need not be co-Hopfian, this sublattice is generally not complete.

(d) Similar to **Th. 27** [13]. □

We could ask the following

Question. Is H^n solitary in the direct product G^n whenever H is solitary in G?

The answer is negative and the next proposition clarifies this.

Proposition 16. *Let H be a solitary subgroup of G. Then the following conditions are equivalent: (1) H^2 is a solitary subgroup of G^2.*

(2) H^n is a solitary subgroup of G^n for every number $n \in \mathbb{N}$.

(3) H is a strongly invariant subgroup of G.

Proof. (1) $\Rightarrow$ (3) Since solitary subgroups are strictly invariant, from Proposition 15 in [3], H is a strongly invariant subgroup of G.

(3) $\Rightarrow$ (2) Since H is strongly invariant in G then H^n is strongly invariant in G^n for any finite n [6], **SI6**. Let $G^n = G_1 \oplus \cdots \oplus G_n$, where $G_i \cong G$, $\pi_i : G \to G_i$ are the projections, $i = 1, \ldots, n$, and $\varphi : H^n \to K$ is an isomorphism, $K \leq G^n$. Then $K \leq \pi_1 K + \cdots + \pi_n K$ and since H^n is a strongly invariant in G^n, we have $\pi_i \varphi(H^n) = \pi_i K \leq H^n$ for all $i = 1, \ldots, n$, i.e., $K \leq \pi_1 K + \cdots + \pi_n K \leq H^n$. Hence H^n is a solitary in G^n.

(2) $\Rightarrow$ (1) Obvious. □

Example. *Intersections of solitary subgroups need not be solitary.*

Let $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p)$. Then the socle $\mathbb{Z}(p) \oplus \mathbb{Z}(p)$ and $\mathbb{Z}(p^\infty)$ are solitary subgroups of G but their intersection is not a solitary subgroup (not even fully invariant).

In what follows **we present the proofs of our results**, already summarized in the Introduction.

In the first direction, *a special class of groups is considered* (e.g., divisible or torsion-free or torsion or mixed) *and we determine all solitary subgroups*.

Using results recalled in the previous sections we can easily determine the *solitary subgroups of the torsion-free groups*.

Theorem 17. *Let G be a torsion-free group.*

(1) If G is reduced then 0 is the only solitary subgroup of G.

(2) If G is divisible then G is solitary simple iff G has finite rank.

(3) If G is not reduced then the only solitary subgroup of G is $D(G)$, the divisible part, if it is of finite rank.

Proof. Since solitary subgroups must be co-Hopfian, these are divisible, so direct sums of $\mathbb{Q}$. These cannot consist of infinitely many copies (otherwise not co-Hopfian) and cannot have less than the whole divisible part. It remains to notice that $\mathbb{Q}^n$ is strongly invariant in $\mathbb{Q}^n \oplus R$ for any reduced torsion-free group $R \neq 0$. □

Further, the determination of *solitary subgroups of torsion groups* is harder, despite the fact that the standard reduction to primary components works.

Proposition 18. *Let H be a torsion subgroup of a group G. Then H is solitary in G iff each p-component H_p is solitary in G_p.*

Proof. Direct. For an arbitrary fixed prime number p, let K_p be a subgroup of G_p with $H_p \cong K_p$. Then $H = \bigoplus_{q\in \mathbf{P}} H_q \cong \left(\bigoplus_{q\neq p} H_q\right) \oplus K_p$ and since H is solitary in G, $\bigoplus_{q\in \mathbf{P}} H_q = \left(\bigoplus_{q\neq p} H_q\right) \oplus K_p$. Hence $H_p = K_p$ (by uniqueness of primary components as subgroups). Conversely, let K be a subgroup of G with $H \cong K$. Then $\bigoplus_{q\in \mathbf{P}} H_q \cong \bigoplus_{q\in \mathbf{P}} K_q$ and so $H_q \cong K_q$ for every prime number q (isomorphic torsion groups have isomorphic primary components). Hence, by hypothesis, $H_q = K_q$ for every q and $H = K$, as desired.

Alternatively, since a torsion group is co-Hopfian iff its primary components are co-Hopfian, and, every torsion strictly invariant subgroup of any group is strongly invariant, and, a subgroup H is strongly invariant in a torsion group G iff for every prime p, H_p is strongly invariant in G_p, the claim follows. □

We recall from [3], the following

Theorem 19. *Let H be a subgroup of a p-group G. Then H is strictly invariant iff it satisfies one of the following conditions.*

1. *$H = G$.*
2. *there exists a non-negative integer n such that $H = G[p^n]$.*
3. *there exists a non-negative integer n such that $H = G[p^n] + D(G)$.*

Therefore we may proceed with p-groups, and, due to Corollary 7, (b), we have to find the solitary subgroups of direct sums $\mathbb{Z}(p^\infty)^n \oplus R$ with co-Hopfian reduced R, i.e., to find the solitary subgroups of a reduced p-group. More precisely, if H ranges over all the solitary subgroups of R then $\mathbb{Z}(p^\infty)^n \oplus H$ are the only solitary subgroups of $\mathbb{Z}(p^\infty)^n \oplus R$.

Theorem 20. *Let H be a subgroup of a reduced p-group G. Then H is solitary in G iff $H = G[p^n]$ for some positive integer n and H is finite or else $H = G$ and* $\operatorname{card} H = 2^{\aleph_0}$.

Proof. To prove this characterization we just use the following ingredients:

the torsion strictly invariant subgroups are strongly invariant,

for an arbitrary p-group, the subgroups $G[p^n]$ are strongly – and so also strictly – invariant,

the only (reduced) strongly invariant subgroups of a reduced p-group are the subgroups $G[p^n]$ for all positive integers n.

The cardinality part follows from Corollary 11 and the theorem before: any reduced countable co-Hopfian p-group is finite (and finite groups are co-Hopfian) and there are no infinite reduced co-Hopfian p-groups G such that $\operatorname{card} G = \aleph_0$ or $\operatorname{card} G > 2^{\aleph_0}$. □

The determination of the *solitary subgroups for mixed groups*.

Theorem 21. *The solitary subgroups of a splitting reduced mixed group G are the solitary subgroups of the torsion part if the* 2*-component of G, $G_2 = \{a \in G : \exists n \geq 0, a^{2^n} = 0\} = 0$.*

Proof. Consider a splitting mixed group $G = T(G) \oplus F$. The condition $G_2 = 0$ is necessary in order to have only splitting subgroups for G. Indeed, if $G_2 = 0$ then all strictly invariant subgroups are projective invariant (see [3]). Therefore if $H = T(H) \oplus K$ is a solitary subgroup, it is co-Hopfian and so K is co-Hopfian torsion-free. Since G is reduced, K is reduced and we have $K = 0$. □

If the torsion part $T(G)$ is bounded (and the 2-component is zero), according to Baer, Fomin theorem (**100.1** in [10]), the group is splitting and the above theorem applies.

Next we follow the second direction: *for a* (more or less) *arbitrary group we determine solitary subgroups which belong to a special class* (i.e., divisible subgroups, p-subgroups and mixed subgroups because this is already clear for torsion-free subgroups).

The solitary divisible subgroups are determined in the following

Proposition 22. *(i) The only solitary divisible p-subgroup of a group G is the (whole) divisible p-part, i.e., $D(G_p)$, if it is of finite rank.*

(ii) The only solitary divisible torsion-free subgroup of a group is the (whole) divisible torsion-free part, if it is of finite rank.

Proof. If such divisible subgroups have infinite rank, these are not co-Hopfian (nor solitary). If less than the whole divisible part, these are not strictly invariant (nor solitary). Conversely, having finite rank, these divisible parts are co-Hopfian. These are also strictly invariant, because as fully invariant direct summands, such subgroups are strongly invariant (see [6]). □

Moreover

Corollary 23. *The divisible part $D(G)$ of a group G is solitary, iff $D_p(G)$ is of finite rank for all primes p and $r_0(D(G))$ (the torsion-free rank of $D(G)$) is finite.*

Proof. Just use Propositions 18, 15 (c) and 22. □

Useful for the standard reductions is the following

Proposition 24. *Let H be a subgroup of $G = D(G) \oplus R$ with divisible part $D(G)$ and reduced direct complement R. If $H = D(H) \oplus L$ with L chosen as a subgroup of R then H is solitary in G iff $D(H)$ is solitary in $D(G)$, L is solitary in R and $S_L(D(G)) \leq D(H)$.*

Proof. Proposition 15 from [3] shows that the conditions are necessary. These are also sufficient. Indeed, since $\mathrm{Hom}(D(H), R) = 0$, every monomorphism $\varphi : H \to G$ may be presented as a matrix $\varphi = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$, where $\alpha : D(H) \to D(G)$ is a monomorphism and $\beta : L \to D(G)$, $\gamma : L \to R$. Note that γ is a monomorphism. Indeed, if $0 \neq x \in \ker \gamma$ then $\beta(x) \neq 0$ (since φ is a monomorphism). Since $\beta(x) \in D(H) = \mathrm{im}\, \alpha$, we deduce $\beta(x) = \alpha(y)$ for some $y \in D(H)$ and $0 \neq x - y \in \ker\varphi$, a contradiction. Therefore $\gamma(L) = L$, whence $\varphi(H) = H$. □

Note that L cannot always be chosen for $L \leq R$, but it can be chosen if the 2-component vanishes (as in Theorem 21).

As a special case, *the socle of a p-group G is solitary iff G is of finite rank*. Moreover, *the socle of a torsion group is solitary exactly if all primary components are of finite rank*.

Since the torsion part $T(G)$ of an Abelian group is obviously (strongly and so) strictly invariant in G, from Theorem 0 (introduction) we obtain

Proposition 25. *$T(G)$ is a solitary subgroup of a group G if and only if $T(G)$ is co-Hopfian (as a group).*

Therefore, fully invariant (pure) subgroups need not be solitary. A simple example is pG for $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}(p^i)$. Since $pG \cong G$, this is a fully invariant subgroup which is not strictly invariant.

Neither are fully invariant direct summands. The question reduces (modulo co-Hopfian) to strictly invariant subgroups, and was discussed in the previous Section.

Further

Proposition 26. *The p-components of a torsion group G are solitary subgroups iff G is co-Hopfian.*

Proof. The p-components are uniquely determined by G with respect to the property of being p-groups with direct sum $= G$. Therefore, for a prime number p, $H \cong G_p$ does (generally imply $H \leq G_p$, i.e., G_p are strictly invariant, but does) not generally imply $H = G_p$ unless G_p is co-Hopfian. □

Finally, a similar proof to the one of Proposition 24 shows that

Proposition 27. *If $G = T(G) \oplus F$ with reduced torsion part $T(G)$ and torsion-free direct complement F, T is solitary in $T(G)$, $S_K(T(G)) \leq T$ and K is solitary in F then $T \oplus K$ is solitary in G.*

Corollary 28. *If $G = T(G) \oplus F$ with reduced torsion part $T(G)$ and torsion-free direct complement F, T is solitary in $T(G)$ and K is solitary in F then $T \oplus K$ is solitary in G.*

Proof. Follows from the previous proposition since $\mathrm{Hom}(K, T(G)) = 0$ if we note that K is divisible (Theorem 17) and $T(G)$ is reduced. □

By Corollary 7, any split (genuine) mixed co-Hopfian group is not solitary simple, since its torsion part is a solitary subgroup. From [3], Theorem 20 it follows that if the p-component of a solitary subgroup F of a reduced group G is unbounded then $T_p(G) \leq F$.

We end this section with an

Example of a *solitary genuine mixed subgroup of a co-Hopfian group*.

Let H be a maximal pure subgroup of the group of p-adic integers J_p containing $\mathbb{Z}_p$. Then H has cardinality $2^{\aleph_0}$ and $J_p/H \cong \mathbb{Q}$. Let G be the mixed group from [11], Theorem 3.1, with $T(G) = T$, a semi-standard reduced p-group, $G/T \cong H$ and $F = pG + T$. Since $F/T = (pG+T)/T = p(G/T) \cong H$ and $F/p\mathbb{Z}_p \cong A$, where $A = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}; T)$ (see [11], § 3), it follows that, as G is, F is also co-Hopfian.

We shall show that F is a solitary in G. Assume that $f : F \to G$ is a monomorphism with $f(F) \nleq F$. Since $f(T) \leq T$ we have $f(F/T) \nleq F/T$ and so $f(p\mathbb{Z}_p) \nleq F$. Hereinafter $pf(F) \leq F$ and $T[p] = \ker pf$. Consider pf as endomorphism of F/T. In the same way, as in [11], the action of pf reduces to multiplication with some rational number $\frac{n}{m}$, where n, m are coprime and $p \nmid m$. Thus, $mpf - n1_F$ induces the zero map on F/T and so $(mpf - n1_F)F \leq T$, in particular $(mpf - n1_F)(p\mathbb{Z}_p) \leq T$. However, every homomorphic image of $p\mathbb{Z}_p$ in T is cyclic, and so bounded, so there exists an integer $r \geq 0$ such that $p^r(mpf - n1_F)(p\mathbb{Z}_p) = 0$. Thus, the endomorphism $p^r(mpf - n1_F)$ of F annihilates $p\mathbb{Z}_p$ and so passes to the quotient inducing a map: $F/p\mathbb{Z}_p \cong A \to T$. This image is both cotorsion and torsion, and hence is bounded. So replacing r by a larger integer if necessary, we may suppose that this image is zero i.e. $p^r(mpf - n1_F) = 0$. Since T is unbounded there is a cyclic summand $\langle x\rangle$ of T of order greater than p^r, say $\mathrm{ord}(x) = r + s$. Thus, $0 \neq p^r m(p^{s-1}x) \in T[p]$ and so $p^r n(p^{s-1}x) = pf(p^r m(p^{s-1}x)) = 0$ since $\ker pf = T[p]$. Hence, p divides n, say $n = pn'$ and so $p^{r+1}f = p^r\frac{n}{m}$. Finally $p^{r+1}f(F/T) = p^{r+1}(n'/m)(F/T)$, whence $f(F/T) = (n'/m)(F/T) = F/T$, since F/T is torsion-free and n'/m induces an automorphism of F/T, which is a contradiction.

5. Two extreme classes

Obviously, 0 *is solitary in every group* G. A subgroup H will be termed *proper* in G if $0 \neq H \neq G$. A group G is called *solitary simple* (or solitary-*free* in [13]) if 0 and G are its only solitary subgroups (equivalently, if it has no proper solitary subgroups). According to Corollary 1, *only co-Hopfian groups may be solitary simple*.

Again it is easier to dispose of *the torsion-free case*.

Using Propositions 8 and 22, (ii) we deduce

Theorem 29. *The only solitary simple torsion-free groups are the finite rank divisible groups.*

Proof. Since any solitary subgroup H of a group G is co-Hopfian as a group, if it is torsion-free it must be divisible, so is a direct summand of G. So $H = G$ since otherwise H is not strictly invariant in G. □

Corollary 30. *A divisible group $D \neq 0$ is solitary simple iff it is of finite rank, torsion-free or p-group.*

This can be rephrased as

Proposition 31. *A divisible group $D \neq 0$ has proper solitary subgroups iff the rank of D_p is nonzero finite and $D_p \neq D$ at least for one prime p.*

Proof. If a solitary subgroup H is not torsion then by Corollary 19 in [3] , $H = D$. So if $0 \neq H \neq G$ then H is torsion and H is solitary in $T(D)$. Hence the rank of D_p is finite at least for one prime p. The sufficiency is obvious. □

Further, we determine *the torsion solitary simple groups.*

From Proposition 26 it follows that co-Hopfian torsion groups are not solitary simple unless these are p-groups (otherwise the p-components are proper solitary subgroups). Therefore, we just have to find the *co-Hopfian p-groups* which are solitary simple.

Since we already dealt with divisible solitary simple groups, we go into two cases.

(a) $G = D(G) \oplus R$ with $D(G) \neq 0 \neq R$ reduced, is a co-Hopfian p-group. Such groups are not solitary simple since $D(G)$ is a proper solitary subgroup of G.

(b) G is a reduced co-Hopfian p-group. By Corollary 11(a), such a p-group is either finite or else of the power of continuum.

Finite solitary simple (not necessarily commutative) p-groups were determined in [13].

We mention here that in the Kaplan, Levy's paper (submitted December 2007), the authors claim their intention to deal with solitary subgroups of Abelian groups ("work in preparation" in References), but this did not happen the next 13 years (though both authors published meanwhile together or separately at least 16 papers each).

For the sake of completeness, *we quote* from [13] the following (in the paragraph below, groups may not be commutative and so the group operation is multiplication).

Definition. A nontrivial group is called *(normal) solitary-free* if it has no proper (normal) solitary subgroups.

Note that all groups which are direct products of copies of a simple group T are (normal) solitary-free. This is an immediate consequence of the fact that such groups are characteristically simple. We shall call (normal) solitary-free groups which are not of the form T^n, where T is simple, *special (normal) solitary-free* groups. In [13] Section 2 we can find

Proposition 32. *A special solitary-free p-group must be non-Abelian.*

Proof. Let G be an abelian p-group. Then $\Omega_1(G) = \{x \in G : x^p = 1\}$ is a nontrivial solitary subgroup of G which is isomorphic to $\mathbb{Z}_p^n$ for some positive integer n. If G is solitary-free, then $G = \Omega_1(G)$ which is not special. □

Therefore, for our paper, we immediately derive

Corollary 33. *A finite (Abelian) p-group is solitary simple iff it is finite rank elementary.*

Proof. By the previous proposition, such a p-group is special, so a finite direct sum of simple p-groups. Hence it is isomorphic to $\mathbb{Z}(p)^n$ for some positive integer n, i.e. a finite rank elementary p-group. The converse is obvious, but more can be shown (using Proposition 25 in [6]): a torsion group is strictly invariant simple iff it is an elementary p-group (i.e., iff it is fully invariant simple). □

Remark. Again in the not necessary commutative case, for completeness, we mention that *a finite solvable group is characteristically simple iff it is an elementary Abelian group*.

Finally some easy remarks about (genuine) *mixed solitary simple* groups.

According to Proposition 25, a genuine mixed group with *co-Hopfian torsion part* is *not solitary simple*.

By corollary 3 any split (genuine) mixed co-Hopfian group is not solitary simple. So if there exist (genuine) mixed groups G which are solitary simple, these are not splitting, every p-component has finite Ulm invariants.

In the light of the example which closes the previous section, we ask the following

Question. Do co-Hopfian mixed solitary simple groups exist?

Remark. Another, more general definition for "solitary simple" would be: a group G whose only solitary subgroup is 0. According to Corollary 1, these are not co-Hopfian. We do not address the determination of such groups in this paper.

In closing, here are the groups all whose subgroups are solitary.

Theorem 34. *All the subgroups of an Abelian group are solitary iff the group is a direct sum of cocyclic p-groups, at most one for each prime number p.*

Proof. From Theorem 0 (introduction), such a group must be co-Hopfian, and so are all its subgroups. Since $\mathbb{Z}$ is not co-Hopfian, such a group must be torsion. According to Proposition 18, a torsion group has this property iff all its p-components have this property. For any fixed prime number p, let G be a co-Hopfian p-group all whose subgroups are (co-Hopfian and) solitary. If G contains a subgroup H which is the direct sum of two of its proper subgroups, the socle of H contains at least two nonisomorphic order p subgroups, none of which is solitary. Therefore the rank of G must be 1. However, this happens exactly when G is isomorphic to a subgroup of $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$. Since the first case cannot happen, finally G is cocyclic, finite or infinite. Finally, cocyclic p-groups have a subgroup lattice which is a chain, and so clearly all subgroups are solitary. □

This result is just a special case of

Theorem 35 (*[19]*)**.** *Over a commutative Noetherian ring R, all submodules of a module are solitary exactly if it is isomorphic to a submodule of the injective hull E of the module $L(R/M_\alpha)$, where M_α runs over the set of maximal ideals of R.*

As a special case, we obtain the commutative case of the following result (from [13])

Theorem 36. *Let G be a finite group. Then, every cyclic subgroup of G is solitary in G if and only if G is cyclic.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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