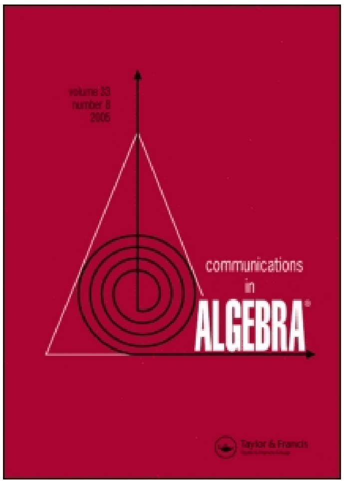

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Grigore Clugreanu [a]
[a] Dept. of Math. and Computer Sci., Babes-Bolyai University, Cluj-Napoca, Romania

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ABELIAN GROUPS WITH SEMI-LOCAL ENDOMORPHISM RING

Grigore Călugăreanu

Dept. of Math. and Computer Sci.,
Babes-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail: calu@math.ubbcluj.ro

ABSTRACT

The abelian groups with semi-local endomorphism ring are characterized, with one exception: the infinite rank torsion-free ones.

A ring R is called *semi-local* if $R/\text{rad}\, R$ is semisimple artinian.

In his authoritative book on infinite abelian groups, Laszlo Fuchs, the leading expert on this topic, asks: ''For which abelian groups is the endomorphism ring semi-local?'' ([4], Problem 84). This problem has been open for 26 years. Modules whose endomorphism ring is semi-local have been investigated by several authors (see, e.g.,[3] and the literature listed there).

The main result of this paper is the following theorem:

Theorem. *Let G be an abelian group with endomorphism ring* $\text{End}(G)$ *and torsion subgroup* $T(G)$. *Then:*

- $\text{End}(G)$ *is semi-local if and only if* $T(G)$ *is finitely generated and G is a direct sum of form* $T(G) \oplus F$, *where F is a torsion-free subgroup of G such that* $\text{End}(F)$ *is semi-local.*
- *If G is divisible, then* $\text{End}(G)$ *is semi-local if and only if G has finite rank.*

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- *If G is torsion-free reduced of finite rank, then* End(G) *is semi-local if and only if* $pG = G$*, for all but finitely many prime integers* p.

1 PRELIMINARY RESULTS

In this note, "group" will mean "abelian group". For unexplained terminology and facts, we refer to Fuchs[4]. We denote by $r(G)$ the rank (the Goldie dimension) of a group G.

We start our discussion with a few lemmas on groups with semi-local endomorphism ring. Some of the next results may be known, but we include some of proofs for the sake of completeness.

For an artinian ring R, if $\mathcal{E}$ denotes the set of all finite subsets of pairwise orthogonal idempotents whose sum is 1, notice that $\{\mathrm{card}(E) | E \in \mathcal{E}\}$ is bounded. It is also bounded for each semi-local ring (non-zero orthogonal idempotents do not collapse modulo $J = \mathrm{rad}(R)$, the Jacobson radical of the ring R).

For easy reference we state the following lemma:

Lemma 1.1. *If G is a group whose endomorphism ring* End(G) *is semi-local, there is a positive integer m such that no direct decomposition of G has more than m summands.* □

Lemma 1.2. *If the endomorphism ring* End(G) *of a group G is semi-local, then every direct summand H of G has semi-local endomorphism ring.*

Proof. If e is idempotent in a semi-local ring R, then eRe is also semi-local. As a special case, if $G = H \oplus K$ and π: $G \to H$ denotes the corresponding projection, we can identify $\mathrm{End}(H)$ with $e\mathrm{End}(G)e$, where $e = j \cdot \pi$ with j: $H \to G$ the inclusion. □

The following lemma is well known, and so the proof is omitted.

Lemma 1.3. *Let A, B be rings,* ${}_AC_B$ *be a bimodule and* $R = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

Then $\mathrm{rad}(R) = \begin{pmatrix} \mathrm{rad}(A) & C \\ 0 & \mathrm{rad}(B) \end{pmatrix}$. *Moreover,*

$$R/\mathrm{rad}(R) \simeq \begin{pmatrix} A/\mathrm{rad}(A) & 0 \\ 0 & B/\mathrm{rad}(B) \end{pmatrix} \simeq A/\mathrm{rad}(A) \oplus B/\mathrm{rad}(B).$$ □

Corollary 1.1. *In the setting of the previous lemma, if A and B are semi-local, so is R.* □

Corollary 1.2. *If $G = H \oplus K$, H is a fully invariant subgroup of G and* $\mathrm{End}(H)$ *and* $\mathrm{End}(K)$ *are semi-local, then* $\mathrm{End}(G)$ *is also semi-local.*

Proof. Indeed, in this case $\mathrm{Hom}(H, K) = 0$, $\mathrm{End}(G) \simeq \begin{pmatrix} \mathrm{End}(H) & \mathrm{Hom}(K,H) \\ 0 & \mathrm{End}(K) \end{pmatrix}$ and the previous corollary applies. □

Corollary 1.3. *If $G = H \oplus K$, where K is a finite group, H is a finite rank divisible group, and* $\mathrm{End}(H)$ *and* $\mathrm{End}(K)$ *are semi-local, then* $\mathrm{End}(G)$ *is also semi-local.* □

Lemma 1.4. *If* $\mathrm{End}(G)$ *is semi-local, then* $\mathrm{End}(G^n)$ *is semi-local for every positive integer n.*

Proof. The endomorphism ring $\mathrm{End}(G^n) \simeq \mathcal{M}_n(\mathrm{End}(G))$, the full matrix ring, and consequently $\mathrm{End}(G^n)/J(\mathrm{End}(G^n)) \simeq \mathcal{M}_n(\mathrm{End}(G)/J(\mathrm{End}(G))$ is semi-simple. □

Lemma 1.5. *If $G = \oplus_{i=1}^{n} H_i$ is a (finite) direct sum of fully invariant subgroups whose endomorphism rings* $\mathrm{End}(H_i)$ *are semi-local, then* $\mathrm{End}(G)$ *is also semi-local.*

Proof. This is an immediate consequence of $\mathrm{End}(G) \simeq \prod_{i=1}^{n} \mathrm{End}(H_i)$ and of the fact that finite direct products of semi-local rings are also semi-local. □

It is easy to dispose of the mixed case by reducing the problem to the torsion and torsion-free cases.

Proposition 1.1. *A mixed group G has semi-local endomorphism ring if and only if $G = T \oplus F$ with T torsion and F torsion-free, both with semi-local endomorphism ring.*

Proof. By Lemma 1.1, there is a positive integer m such that every decomposition of G has at most m direct summands. Observe that in this case its torsion part $T(G)$ has the same property.

To see this, suppose $T(G) = A_1 + \cdots + A_m + A_{m+1}$ is a direct sum of $m + 1$ non-zero subgroups; then in each of these torsion groups A_i, one can find a direct summand C_i which is either a cyclic p-group or quasi-cyclic p-group for some prime p (depending on A_i). Then $C_1 + \cdots + C_m + C_{m+1}$, being a direct summand of $T(G)$, is a sum of a bounded pure subgroup and a divisible subgroup of G and hence a direct summand of G. This gives a direct decomposition of G with more than m direct summands.

Consequently, the subgroups of T satisfy the minimum condition, and so T has finite rank. Therefore T is a direct sum of a bounded group and a torsion divisible group. Hence G is splitting, as guaranteed by a classical result of Baer and Fomin [Theorem 100.1, chap XIV[4]]. The rest follows from Lemma 1.2.

Conversely, the use of the isomorphism

$\mathrm{End}(G) \simeq \begin{pmatrix} \mathrm{End}(T) & \mathrm{Hom}(F,T) \\ 0 & \mathrm{End}(F) \end{pmatrix}$ along with Corollary 1.1 completes the proof. □

2 THE TORSION CASE

As is often the case in the theory of abelian groups, the discussion of torsion groups can be reduced to the case of *p*-groups.

Proposition 2.1. *The endomorphism ring of an infinite rank p-group is not semi-local.*

Proof. A *p*-group G has finite rank exactly if it satisfies the minimum condition on subgroups, or, equivalently, if there is a positive integer m such that no subset of pairwise orthogonal idempotents of $\mathrm{End}(G)$ whose sum is 1 has more than m elements.

Due to Lemma 1.1 we conclude that for an infinite rank *p*-group G, $\mathrm{End}(G)$ is not semi-local. □

It should be emphasized that actually the number m above is the length of the semisimple ring $\mathrm{End}(G)/\mathrm{rad}(\mathrm{End}(G))$.

As a matter of fact, it is not difficult to give examples of torsion groups whose endomorphism ring is not semi-local: if P is an infinite set of prime numbers, just take $A = \oplus_{p\in P}\mathbf{Z}(p)$.

Theorem 2.1. *The endomorphism ring of a torsion group is semi-local if and only if the group has finite rank*.

Proof. Let G be a torsion group whose endomorphism ring is semi-local. As the *p*-components of a torsion group are fully invariant subgroups, in view of Lemma 1.5 and the previous proposition, only the sufficiency requires a proof. Although this could be obtained in more generality (for an artinian right R-module M the endomorphism ring $\mathrm{End}_R M$ is semi-local[1]), we give here a direct proof.

If G has finite rank, it is a finite direct sum of cocyclic subgroups (each having a local ring of endomorphisms). To conclude that $\mathrm{End}(G)$ is semi-local we only have to use Lemma 1.4, Lemma 1.5, Corollary 1.2 and the Corollary 1.3. □

3 THE FINITE RANK TORSION-FREE CASE

It is considerably more difficult to characterize torsion-free groups that have semi-local endomorphism rings. We do not have a characterization in general, but a fairly informative result is available for groups of finite rank.

We begin with some common reductions.

Proposition 3.1. *A divisible group has semi-local endomorphism ring if and only if the group has finite rank.*

Proof. By Lemma 1.1, the endomorphism ring of an infinite direct sum of groups is not semi-local. Thus a direct decomposition of such a divisible group contains only finitely many copies of $\mathbf{Z}(p^{\infty})$ and $\mathbf{Q}$.

Conversely, clearly $\mathrm{End}(\oplus_{i=1}^{n}\mathbf{Q}) \simeq \mathcal{M}_n(\mathbf{Q})$, the full matrix ring with rational entries, is simple, thus artinian. Therefore the proof is completed using Lemma 1.4 and Lemma 1.5 (for the quasicyclic summands). □

Proposition 3.2. *Let $G = D \oplus R$ be a direct sum of a divisible group D and a reduced group R. Then* End(G) *is semi-local if and only if both* End(D) *and* End(R) *are semi-local.*

Proof. Indeed, since $G = D \oplus R$ with D a divisible group and R a reduced group, it is immediate that $\mathrm{Hom}(D, R) = 0$, and so $\mathrm{End}(G) \simeq \begin{pmatrix} \mathrm{End}(D) & \mathrm{Hom}(R,D) \\ 0 & \mathrm{End}(R) \end{pmatrix}$. Using Corollary 1.1 and Lemma 1.2 we obtain at once End(G) semi-local, as desired. □

In the following result we give a characterization of finite rank reduced torsion-free groups with semi-local endomorphism ring.

Theorem 3.1. *Let G be a finite rank reduced torsion-free group. The endomorphism ring* End(G) *is semi-local if and only if $pG = G$, for all but finitely many prime integers p.*

Proof. If $E = \mathrm{End}(G)$ is a semi-local ring, it has only finitely many maximal (two-sided) ideals.

For an arbitrary prime number p consider the ideal pE of E. If $pE < E$ then pE is contained in a maximal ideal $pE \leq M < E$. Notice that when p and q are distinct prime numbers, no maximal ideal M contains both pE and qE (otherwise, $pE + qE = E$ would imply $M = E$). In view of the previous remark, $pE = E$ must hold for almost all prime numbers p. But $pE = E$ is equivalent to $pG = G$ (use $1_G \in E$ and the fact that G is torsion-free), and consequently $pG = G$ holds, for all but finitely many prime numbers p.

For the converse, first observe that E has finite rank if G has finite rank.

To conclude that E/J is semisimple artinian we shall verify that the Jacobson radical of E can be represented as finite intersection of maximal left ideals.

We split the maximal left ideals of E into two classes:

(i) Maximal left ideals between pE and E, for the finitely many prime numbers p allowed by our hypothesis.

Since E has finite rank (as an abelian group) there are only finitely many such maximal left ideals (because E/pE is finite).

(ii) Maximal left ideals not in the previous class.

Let L be a maximal left ideal as in (ii). Since E/L is a simple E-module, it must be torsion-free divisible as an abelian group and of finite rank. Set $\bar{J} = L_1 \cap \cdots \cap L_m$ and represent the radical $J = \bar{J} \cap L \cap L' \cap \cdots = L_1 \cap \cdots \cap L_m \cap L \cap L' \cap \cdots$, where $L_1, \ldots, L_m$ are the maximal left ideals of class (i) and $L, L', \ldots$ are the maximal left ideals of class (ii).

Suppose $\bar{J} > \bar{J} \cap L > \bar{J} \cap L \cap L' > \cdots$ Then $r(\bar{J}) \geq r(\bar{J} \cap L) + 1$, $r(\bar{J} \cap L) \geq r(\bar{J} \cap L \cap L') + 1, \ldots$ and $r(E/\bar{J}) < r(E/(\bar{J} \cap L) < r(E/(\bar{J} \cap L \cap L') < \cdots$ Since the rank of E is finite, the sequence $\bar{J} \geq \bar{J} \cap L \geq \bar{J} \cap L \cap L' \geq \cdots$ has to stop after finitely many steps. Thus, the Jacobson radical can be represented as a finite intersection of maximal left ideals.

Consequently, E/J is semi-simple artinian and E is semi-local. □

Remarks. Using the characterization of the semi-local rings given in[1], another proof of the second part of the previous theorem is available: if n is the finite rank of a torsion-free group G and $pG = G$ holds for all but finitely many prime numbers p, then it is not hard to see that the mapping $d : \mathrm{End}(G) \to \{0, 1, \ldots, n\}$, which assigns to each endomorphism $f \in \mathrm{End}(G)$ the torsion-free rank $r_0(\mathrm{co}\ker f)$, satisfies Camps and Dicks' conditions from[1], and so $\mathrm{End}(G)$ is semi-local.

Notice that our result generalizes Corollary 3 of[5]. All the groups with semi-local endomorphism ring have the cancellation (and so also the substitution) property. So far we do not see a direct link between these two properties.

Examples. Finite rank torsion-free groups with semi-local endomorphism ring are abundant. Take a finite algebraic extension A of $\mathbf{Z}$ and a finite number of localizations A_p of A corresponding to prime numbers p of $\mathbf{Z}$. Then the intersection $H = \cap_{i=1}^{n} A_{p_i}$ is a semi-local ring. Using a well-known result of Corner[2], there is a finite rank torsion-free group G such that $\mathrm{End}(G) \simeq H$.

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