Self-dual notions for groups

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Abstract

With respect to socle and radical we can define dual notions for fully invariant, characteristic respectively normal subgroups of groups. These turn out to coincide with the initial notions.

1 The "dual" notions

Introduced in 1933 by F. Levi for groups, the role of the fully invariant subobjects in several algebra categories (and especially for groups) is well-known. A subgroup H of a group G was called *strongly invariant* in G (see [1] and also [2]), if $f(H) \leq H$ for every group homomorphism $f : N \longrightarrow G$. Recently, a study of such subgroups was initiated by the author, mainly in the case of Abelian groups (see [1]).

A useful notion (merely used for modules and Abelian groups, as a special case) is the following

Definition. For any pair of groups H, G, the *H-socle* of G is $S_H(G) = \langle \bigcup \{\mathrm{im} f | f : H \longrightarrow G\} \rangle$ $(= \sum \{\mathrm{im} f | f : H \longrightarrow G\}$, for Abelian groups).

An immediate application of this construction is

Lemma 1 *A subgroup H of a group G is strongly-invariant iff $S_H(G) = H$.*

Proof. Obviously, $H \leq S_H(G)$ (because H is the image of the inclusion $i : H \longrightarrow G$) and $S_H(G) \leq H$ iff $f(H) \leq H$ for every morphism $f : H \longrightarrow G$. ■

Now consider the dual notion for the H-socle

Definition. For any pair of groups H, G, the *H-radical* of G is $R_H(G) = \bigcap \{\ker f | f : G \longrightarrow H\}$.

Dually, for a normal subgroup K of G, let G/K be a factor group of G. Then $R_{G/K}(G) = \bigcap \{\ker f | f : G \longrightarrow G/K\}$.

Obviously $R_{G/K}(G) \leq K$ (K is the kernel of the canonical projection $p_K : G \longrightarrow G/K$).

Hence, the notion which should be dual to strongly invariant, is:

Definition. A subgroup K of a group G is called *strongly co-invariant* if $R_{G/K}(G) = K$.

Equivalently, for every morphism $f : G \longrightarrow G/K$, $\ker f \geq K$ (i.e., $\ker f \geq \ker p_K$) (i.e., $R_{G/K}(G) \geq K$).

Further, recall that a subgroup H of a group G is *fully invariant* if $\mathrm{im}(f \circ i) \leq \mathrm{im}(i)$ holds for every endomorphism f of G (here i denotes the inclusion).

Finally, the dual notion for fully invariant is given by the following

Definition. A subgroup K of a group G is called *fully co-invariant* if $\ker(p_K \circ f) \geq \ker p_K$ for every endomorphism f of G, i.e., $\ker(p_K \circ f) \geq K$.

From this point of view we can continue with the following dual definitions

A subgroup K of a group G is *co-characteristic* if $\ker(p_K \circ f) \geq \ker p_K$ for every automorphism f of G, i.e., $\ker(p_K \circ f) \geq K$, and *co-normal* if this holds for every inner automorphism of G. This way, fully co-invariant $\Longrightarrow$ co-characteristic $\Longrightarrow$ co-normal.

However these notions turn out to be self-dual

Proposition 2 *A subgroup is fully co-invariant (respectively co-characteristic or co-normal) iff it is fully invariant (respectively characteristic or normal).*

Proof. Immediate from definitions: K is fully co-invariant iff for every endomorphism f of G, $(p_K \circ f)(K) = \{K\}$. But this means $f(k)K = K$ or $f(k) \in K$ for every $k \in K$, which is the usual definition of a fully invariant subgroup. Similar co-characteristic $\equiv$ characteristic, and co-normal $\equiv$ normal. ■

However, *strongly co-invariant subgroups do not coincide with strongly invariant.* Here is an example: consider $G = \mathbf{Z}_2 \oplus \mathbf{Z}_8 = \langle a, b | 2a = 8b = 0 \rangle$. Then the socle $G[2] = \{0, 4b, a, a + 4b\}$ is strongly invariant ([1]). However, it is not strongly co-invariant: for $G/G[2] = \{G[2], b + G[2], 2b + G[2], 3b + G[2]\}$ define $g : G \longrightarrow G/G[2]$ by $g(a) = g(b) = 2b + G[2]$. Then $g(G[2]) = \{0, 2b + G[2]\} \neq G[2]$ that is $\ker(g) \ngeq G[2]$.

References

[1] Călugăreanu G. *Strongly invariant subgroups.* 2012, submitted

[2] http://groupprops.subwiki.org/wiki/Homomorph-containing_subgroup