Schreier lattices

1 Introduction

Because of the fundamental use of normal (and composition) series, most of the Group Theory (text)books present three well-known strongly connected results: the Zassenhaus Lemma (1934), the Schreier Refinement Theorem (1928) and Jordan-Hölder Theorem (1868, 1889).

Some Lattice Theory (text)books present the corresponding generalizations, mainly because of the use of lattices of finite length. Despite the fact that [Zassenhaus] $\Rightarrow$ [Schreier] and [Schreier] $\Rightarrow$ [Jordan-Hölder], none of them does mention Zassenhaus (but in the sequel justice is made!). All these generalizations are proved for modular lattices and so, can be also used for modules.

The purpose of this note is to show that Zassenhaus Lemma holds in more general conditions. Therefore, since automatically Schreier Theorem and its Corollary, Jordan-Hölder Theorem also hold, we finally extend the definition of finite length far beyond modularity.

2 Preliminaries

Definition. A lattice L has *condition (C)* if for every 4 elements $a, b, c, d \in L$, $a \leq b$ and $c \leq d$ imply $(b \wedge d) \wedge ((a \wedge d) \vee c) = (b \wedge d) \wedge ((c \wedge b) \vee a)$.

Exercise 1 *Condition (C) is equivalent to modularity.*

Proof. If the lattice is modular then $(b \wedge d) \wedge ((a \wedge d) \vee c) \overset{\text{mo}}{=} (b \wedge d) \wedge ((d \wedge (a \vee c)) = (b \wedge d) \wedge (a \vee c) = (b \wedge d) \wedge (b \wedge (a \vee c)) \overset{\text{mo}}{=} (b \wedge d) \wedge ((c \wedge b) \vee a)$. Conversely, suppose $a \leq b$ and take $d = b \vee c$. Then $b = b \wedge d$, $a = a \wedge d$ and also $(c \wedge b) \vee a \leq b$, such that $b \wedge (a \vee c) = (b \wedge d) \wedge ((a \wedge d) \vee c) = (b \wedge d) \wedge ((c \wedge b) \vee a) = b \wedge (b \wedge c) \vee a = (b \wedge c) \vee a$. $\blacksquare$

Definitions. Let x/y and s/t be intervals in a lattice L. We say (see [1]) that x/y is *upper transpose* of s/t if there are elements $a, b \in L$ such that $x/y = (a \vee b)/b$ and $s/t = a/(a \wedge b)$. We use the notation $x/y \sim s/t$ for upper transpose and $\overset{-1}{\sim}$ for the inverse relation: *lower transpose*. Notice that x/y is upper transpose of s/t iff $y \vee s = x$ and $y \wedge s = t$. It is readily seen that upper (or lower) transpose relation is a partial order relation (reflexive, transitive and antisymmetric) on the set of all intervals

Two intervals are *transpose* if one of them is upper transpose for the second, and, *projective* if they can be included in a finite sequence of intervals, each successive intervals being transpose. Owing to the transitivity previously mentioned, in such a sequence of transpose intervals, superfluous intervals will be deleted.

Two chains

$$a = a_0 \le a_1 \le ... \le a_m = b$$
$$a = b_0 \le b_1 \le ... \le b_n = b$$

are *equivalent* if $m = n$ and there is a permutation $\sigma \in S_n$ such that the intervals a_i/a_{i-1} and $b_{\sigma(i)}/b_{\sigma(i)-1}$ are projective.

Exercise 2 *1)* $x/y \sim x/t \Rightarrow y = t$.

2) A lattice L *is distributive iff for every intervals in* L, $x/y \sim s/t \overset{-1}{\sim} x/v \Rightarrow y = v$.

Proof. 1) Obvious: $y = y \wedge x = t$.

2) If $x/y \sim s/t \overset{-1}{\sim} x/v$ then $y \vee s = v \vee s = x$ and $y \wedge s = v \wedge s = t$. Hence $y = v$ by a well-known characterization of distributivity. Conversely, suppose $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$. Then $(a \vee c)/a \sim c/(a \wedge c) = c/(b \wedge c) \overset{-1}{\sim} (b \vee c)/b$ and $a = b$. ■

Remark. Notice that x/y projective with x/z does generally not imply $y = z$. However, in a distributive lattice the implication holds. Indeed, this follows from

Proposition 3 *([1]) If in a distributive lattice* L *two intervals* a/b *and* c/d *are projective, then there exists an interval* x/y *that is transpose of both* a/b *and* c/d.

Common refinements for finite chains.

For two finite chains in an arbitrary lattice

$a = a_0 \le a_1 \le ... \le a_m = b$ (1) and

$a = b_0 \le b_1 \le ... \le b_n = b$ (2)

denote by $a_{ij} = (a_i \wedge b_j) \vee b_{j-1}$ respectively $b_{ji} = (b_j \wedge a_i) \vee a_{i-1}$ for each $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$.

Observe that these elements refine the given chains.

Indeed, if $a \le c$, for every b, $a \le (b \wedge c) \vee a \le c$ holds. Hence, $a_{i-1} \le b_{ji} \le a_i$ respectively $b_{j-1} \le a_{ij} \le b_j$, that is

$$a_{i-1} = b_{n,i-1} \le b_{1i} \le b_{2i} \le ... \le b_{ni} = a_i$$
$$b_{j-1} = a_{m,j-1} \le a_{1j} \le a_{2j} \le ... \le a_{mj} = b_j.$$

Globally, from the chains (1) and (2) we obtain the chains

$$\begin{array}{c} a = b_0 = a_{01} \le a_{11} \le a_{21} \le ... \le a_{m1} = b_1 \le a_{12} \le ... \le a_{mn} = b_n = b \ (3) \\ a = a_0 = b_{01} \le b_{11} \le b_{21} \le ... \le b_{n1} = a_1 \le b_{12} \le ... \le b_{nm} = a_m = b \ (4) \end{array}.$$

Since $a_i = b_{ni}$ and $b_j = a_{mj}$, (4) is a refinement of (1) and (3) is a refinement of (2). The refinements have the same number of elements.

3 The pattern

Lemma 4 *(Zassenhaus) If $a \leq b$ and $c \leq d$ then $a \leq a \vee (b \wedge c) \leq a \vee (b \wedge d) \leq b$ and $c \leq c \vee (d \wedge a) \leq c \vee (d \wedge b) \leq d$. Then $(a \vee (b \wedge d))/(a \vee (b \wedge c)) \sim (b \wedge d)/((a \vee (b \wedge c)) \wedge (b \wedge d))$ and $(c \vee (d \wedge b))/(c \vee (d \wedge a)) \sim (b \wedge d)/((c \vee (d \wedge a)) \wedge (b \wedge d))$. If the lattice is modular these intervals are projective.*

Proof. First for the transpose intervals: indeed, denoting $x = a \vee (b \wedge d)$, $y = a \vee (b \wedge c)$, $s = b \wedge d$ and $t = (a \vee (b \wedge c)) \wedge (b \wedge d)$, $y \vee s = (a \vee (b \wedge c)) \wedge (b \wedge d) = a \vee (b \wedge d) = x$ is obvious, respectively $y \wedge s = t$ according to our notations. The second verification is covered by a double symmetry $a \leftrightarrow c, b \leftrightarrow d$.

If the lattice is modular, by Exercise 1, the denominators of the right intervals intervals are equal, so the left ones are projective, i.e.,

$$(a \vee (b \wedge d))/(a \vee (b \wedge c)) \overset{\text{transp}}{\sim} s/(x \wedge s) = s/(x' \wedge s) \overset{\text{transp}}{\sim} (c \vee (d \wedge b))/(c \vee (d \wedge a)),$$

where $x' = c \vee (d \wedge b)$. ■

Remark. As it is easily seen in the display of the Zassenhaus Lemma's proof, $s/(x \wedge s)$ and $s/(x' \wedge s)$ projective (instead of equal) would suffice for the above (classical) proof. This simple observation was the starting point of our considerations.

Theorem 5 *(Schreier) In any modular lattice, two chains with the same top and the same bottom elements have equivalent refinements.*

Proof. Since the lattice is modular, by Zassenhaus Lemma, the intervals $a_{ij}/a_{i-1,j}$ and $b_{ji}/b_{j-1,i}$ are projective (one takes $b_{j-1} \leq b_j$ and $a_{i-1} \leq a_i$ instead of $a \leq b$ resp. $c \leq d$): more precisely

$$(a \vee (b \wedge d))/(a \vee (b \wedge c)) \text{ projective to } (c \vee (d \wedge b))/(c \vee (d \wedge a))$$

becomes

$$\begin{aligned} b_{ji}/b_{j-1,i} &= (a_{i-1} \vee (a_i \wedge b_j))/(a_{i-1} \vee (a_i \wedge b_{j-1})) \\ &\approx (b_{j-1} \vee (b_j \wedge a_i))/(b_{j-1} \vee (b_j \wedge a_{i-1})) = a_{ij}/a_{i-1,j}. \end{aligned}$$

■

Definition.- A chain between two elements a and b is called a *composition chain* if

$$a = a_0 < a_1 < ... < a_n = b$$

has no refinements (i.e. successive intervals are simple $a_k/a_{k-1} = \{a_{k-1}, a_k\}$ for every $1 \leq k \leq n$). The number n is called the *length* of the composition chain.

An immediate consequence of Schreier theorem is now

Theorem 6 *(Jordan-Hölder) In a modular lattice, any two composition chains between the same two elements are equivalent.*

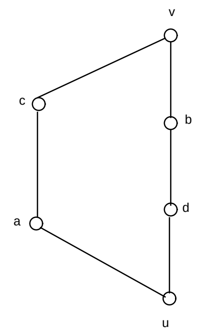
Therefore in the above definition, n is the *length of the interval* b/a, denoted by $l\,(b/a)$. As a special case, in a lattice L with 0 and 1 the *length* $l(L)$ *of the lattice* L is $l\,(1/0)$, if at least one such finite chain between 0 and 1 exists (in this case we say that L has *finite length*).

4 Schreier lattices

A lattice is *Schreier* if every two chains between the same two elements have equivalent refinements.

Problem 7 *Is every Schreier lattice modular? If not, what must be added in order to recover the modularity ?*

Easy example for negative answer of the first question:



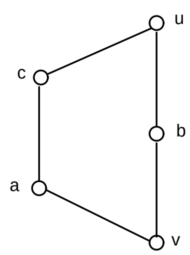
Something similar is the following well-known situation:

[Second isomorphism theorem] *If* a *and* b *are elements in a modular lattice* L *then transposed intervals* $a \vee b/a$ *and* $b/a \wedge b$ *are isomorphic* (in a canonic way).

The converse fails. However, in [1], it was proved:

If a compactly generated lattice L *has the property that* $a \vee b/a \cong b/a \wedge b$ *for every* $a, b \in L$*, then* L *is modular.*

Definitions. Let $\{a, b, c, u, v\}$ be a pentagon in a lattice L (see below).



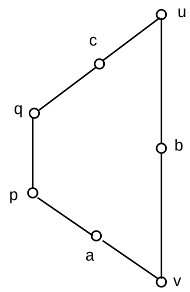
The pentagon is *small* if $a \prec c$ and is *very small* if all the elements cover each other (i.e., the above diagram has no refinements).

Since there are nonequivalent chains from v to u, clearly, *a very small pentagon is not Schreier*.

The existence of a pentagon does **not** imply the existence of a small pentagon, nor of a very small pentagon.

Proposition 8 *In a weakly atomic lattice, if there is a pentagon, there exists also a small pentagon.*

Proof. Indeed, there are elements $a \leq p \prec q \leq c$ which give the small pentagon



We still have $q \| b$ (otherwise $b \leq q$ implies $b \leq c$ respectively $q \leq b$ implies $a \leq b$) and similarly $p \| b$. ■

Example. "finite" ⇒ "compactly generated" ⇒ "weakly atomic".

Question. How to connect the existence of a small pentagon to a very small pentagon ?

Is this continuable ?

References

[1] Crawley P., Dilworth R. *Algebraic Theory of Lattices*. Prentice Hall, Englewood Cliffs, N. J., 1973.