Intersections of principal right ideals in regular rings are principal: a direct proof

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Abstract

Only some indirect proofs are known (using modules or lattices) for a simple ring theoretic statement: in a (von Neumann) regular ring the intersection of two right principal ideals is again a right principal ideal.

In this short note, a direct proof is given and a new property of this lattice is proved.

1 Introduction

In his seminal paper "On regular rings" (1936) [5], von Neumann introduced the (well-known now) new class of rings called *(von Neumann) regular* rings, and (among others) proved that for such rings the set of all the right principal ideals *forms a (modular complemented) lattice (with respect to sum and intersection).*

The existence of the sup's was already proved in von Neumann's paper - and is contained in most of the nowadays textbooks (e.g., see [1] and [2]) - in an elementary way: it suffices to prove that for every two idempotents e and f, the right ideal $I = eR + fR$ is (also) generated by an idempotent. *The plan* is the following:

1) we choose a (suitable) element $a = \overline{e}f$ and show that $eR + fR = eR + aR$;

2) since a is regular, there is an $x \in R$ such that $a = axa$. Then $g = ax$ is an idempotent and the selection of a assures $eg = 0$;

3) we check $aR = gR$;

4) we consider $f' = g\overline{e}$; this is an idempotent, $gR = f'R$ and e, f' are orthogonal;

5) finally, $e + f'$ is an idempotent and $I = eR + fR = eR + aR = eR + gR = eR + f'R = (e + f')R$ which concludes the proof.

Of course, roughly speaking, $eR + fR = (e + \overline{e}fx\overline{e})R$, where $\overline{e}f = \overline{e}fx\overline{e}f$, using regularity for $\overline{e}f$.

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There are currently no direct proofs for the similar inf claim (i.e., in a regular ring, the intersection of two right principal ideals is a right principal ideal). The purpose of this short note is to fill this (74 years old) gap. Before doing this we mention what the reader can find in the existing literature.

In the sequel, all rings R have identity and, for an idempotent e, $\overline{e} = 1 - e$ denotes the complementary idempotent.

2 Von Neumann's proof

In the above mentioned (1936) paper, von Neumann's proof for the existence of the inf's uses the following prerequisites:

1) For any ring R, the set $\mathcal{R}$ of right annihilator ideals in R forms a complete lattice with respect to the partial ordering given by the inclusion, (order) anti-isomorphic to the lattice $\mathcal{L}$ of left annihilator ideals in R (e.g., see [4]).

Actually, the map $\phi : \mathcal{R} \longrightarrow \mathcal{L}$ which gives this anti-isomorphism is simply $\phi(\mathcal{A}) = l(\mathcal{A})$, the left annihilator of $\mathcal{A}$. For right annihilator ideals $\mathcal{A}, \mathcal{B}$, $\inf(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cap \mathcal{B}$ and $\sup(\mathcal{A}, \mathcal{B}) = r(l(\mathcal{A} + \mathcal{B}))$, the right annihilator of the left annihilator of the sum.

2) The following are equivalent for an element $a \in R$:

(i) there is an element $x \in R$ such that $a = axa$;

(ii) there is an idempotent $e = e^2$ such that $aR = eR$, principal right ideals;

(iii) aR has a direct complement (i.e., there is a right ideal $\mathcal{B}$ such that $(aR) + \mathcal{B} = R$ and $(aR) \cap \mathcal{B} = \{0\}$).

3) In a (von Neumann) regular ring, every right principal ideal is the right annihilator of a suitable left principal ideal.

Therefore,

4) In a (von Neumann) regular ring, the above anti-isomorphism ϕ from the principal right ideals of R gives an (order) anti-isomorphism onto the principal left ideals of R.

Clearly, since the definition of a regular ring is right-left symmetric, everything can be stated (and proved) also for left (principal) ideals.

Having recalled all this, *the proof given by von Neumann* was: let $\mathcal{A}, \mathcal{B}$ be principal right ideals of R. The images $\phi(\mathcal{A}) = l(\mathcal{A}), \phi(\mathcal{B}) = l(\mathcal{B})$ are left principal ideals, and so, have a sup which is the sum $l(\mathcal{A}) + l(\mathcal{B})$. Coming back through ϕ^{-1}, the $\inf(\mathcal{A}, \mathcal{B})$ exists and must be $r(l(\mathcal{A}) + l(\mathcal{B}))$. But this is exactly $r(l(\mathcal{A}) \cap r(l(\mathcal{B})) = \mathcal{A} \cap \mathcal{B}$.

3 Lam's proof

A simple and elegant proof was given as **Ex.21.6** [3]: let A, B be two right principal ideals and $C := A + B$. Since R is regular, A, B and C (using the proof for the sup's!) are direct summands in R_R. Thus R/B is a projective

R-module and so is its direct summand C/B. Therefore, the exact sequence

$$0 \longrightarrow A \cap B \longrightarrow A \longrightarrow C/B \longrightarrow 0$$

splits, so $A \cap B$ is a direct summand of A. It follows that $A \cap B$ is a direct summand of R_R, so $A \cap B = eR$ for some idempotent $e \in R$.

Just some comments:
1) first of all, it is kind of frustrating that the proof for the inf's uses the proof of the sup's;
2) in order to understand this proof the reader needs the following (say, graduate) ingredients:

- for a right ideal I the following conditions are equivalent: (a) $I = eR$; (b) I is isomorphic to a direct summand of R; (c) R/I is a projective right R-module;
- direct summands of projective R-modules are projective;
- P_R is projective iff any epimorphism $B_R \longrightarrow P_R$ splits;
- splitting exact sequences are also co-splitting (because above from splitting $A \longrightarrow C/B$ we deduce $A \cap B \longrightarrow A$ is (co-)splitting).

That is, a genuine module machinery for a 100% ring theoretic result!

4 The suitable idempotent

What means a direct (ring theoretic) proof?
Similarly to the sup existence, what is missing (from already 74 years) is, for every two idempotents e and f, an idempotent d such that $eR \cap fR = dR$. Surprisingly enough, the selection of d starts with the same element already mentioned in the Introduction.
In a similar vein with the plan mentioned for the sup in the Introduction:
1) we start with the element $a = e\overline{f}$, which is generally not an idempotent;
2) using the regularity of R, let $x \in R$ be such that $a = axa$; we consider (slightly different) $g = xa$, which is an idempotent;
3) we consider the complementary idempotent $\overline{g}$ and we check $eR \cap fR = fR \cap \overline{g}R$;
4) it can be generally proved that *for idempotents $h, k \in R$, $hkR = hR \cap kR$ if and only if $hk = khk$;*
5) using (4), $fR \cap \overline{g}R = f\overline{g}R$ and so $d = f\overline{g}$ is the required idempotent.

Here is the result together with a simpler proof

Theorem 1 *Let e, f be arbitrary idempotents in a regular ring R and $x \in R$ such that $e\overline{f}xe\overline{f} = \overline{e}f$. Then $d = f - fx\overline{e}f$ is an idempotent and $eR \cap fR = dR$.*

Proof. If $w \in eR \cap fR$, there are elements $u, v \in R$ such that $w = eu = fv$. Hence $ew = e(eu) = eu = w$ and so $\overline{e}w = 0$. We compute $dv = (f - fx\overline{e}f)v = w - fx\overline{e}w = w$ and so $w \in dR$. Conversely, we show that $ed = fd = d$ and so $d \in eR \cap fR$.

First, $fd = f$ is obvious. Secondly, $ed = e(f - fx\overline{e}f) = ef - efx\overline{e}f = ef - (1-\overline{e})fx\overline{e}f = ef - fx\overline{e}f + \overline{e}f = (e+\overline{e})f - fx\overline{e}f = f - fx\overline{e}f = d$. ■

5 Referee is right!

When trying to publish this note the referee claims that the suitable idempotent can be obtained using von Neumann's anti-isomorphism.

Indeed, this can be done (not exactly obvious, using the existing bibliography!) as follows (we continue to use the notations from section two).

Lemma 2 *Let $a, b \in R$ Then $a = ab$ implies $a \in Rb$ which is equivalent to $Ra \subseteq Rb$. If b is an idempotent, the first implication is also an equivalence.*

Proof. Everything straightforward (since $1 \in R$), but: for the converse, if xb then $ab = (xb)b = xb = a$. ■

Further

Proposition 3 *In a regular ring, if e is an idempotent then $l(eR) = R\overline{e}$.*

Proof. Since annihilators of principal ideals are principal ideals (and these may be considered as generated by idempotents), suppose $l(eR) = Rf$. We check $Rf = R\overline{e}$. Since $RfeR = 0$ we deduce $fe = 0$. Therefore $Rf \subseteq R\overline{e}$ $\overset{\text{Lemma}}{\iff} f = f\overline{e} = f(1-e) \iff fe = 0$. Conversely, obviously $R\overline{e} \subseteq l(eR)$ since $R\overline{e}eR = 0$. ■

Now, let us come back to the final paragraph of Section 2.

By the above Proposition, $\phi(eR) = R\overline{e}$ and $\phi(fR) = R\overline{f}$. For their sum, by the well-known formula, $R\overline{f} + R\overline{e} = R(\overline{f} + fx\overline{e}f)$ where $\overline{e}f = \overline{e}fx\overline{e}f$, using regularity for $\overline{e}f$. Then $\phi^{-1}(R(\overline{f} + fx\overline{e}f)) = [1 - (\overline{f} + fx\overline{e}f)]R$ using the left-right symmetric of the previous Proposition. But this is exactly, $R(f - fx\overline{e}f)$ as in our Theorem 1.

6 Application

Our previous result reveals another property of the regular rings.

First recall the following elementary result

Lemma 4 *Let $R_R = I \oplus J$ be a right ideal direct decomposition of R. There exists an idempotent e such that $I = eR$ and $J = \overline{e}R$.*

Much is known about idempotents e and f for which $eR = fR$. For example, here is a selection from Lam [3]:

Proposition 5 *Let e and f be idempotents in a ring R. The following are equivalent:*

1. $eR = fR$
2. $ef = f$ *and* $fe = e$
3. *there exists* $r \in R$ *such that* $f = e + er\overline{e}$
4. *there exists* $u \in U$ *such that* $f = eu$
5. $R\overline{e} = R\overline{f}$.

Next, the direct complements of a right ideal direct summand may be characterized as follows

Lemma 6 *Let $R = eR \oplus gR$. Then $gR = (\overline{e} - er\overline{e})R$ for a suitable $r \in R$. Conversely, for every $r \in R$, $(\overline{e} - er\overline{e})R$ is a direct complement for eR.*

Proof. Since there is an idempotent f such that $eR = fR$ and $gR = \overline{f}R$, according to **3** above, $f = e + er\overline{e}$ and $g = \overline{f} + \overline{f}sf$ for suitable $r, s \in R$. Thus $g = \overline{f}(1 + \overline{f}sf)$ with $u = 1 + \overline{f}sf$ invertible in R (with inverse $1 - \overline{f}sf$). Hence $g = (\overline{e} - er\overline{e})u$ and so $gR = (\overline{e} - er\overline{e})R$.

As for the converse, $eR \oplus (\overline{e} - er\overline{e})R = R$ if there is an idempotent $f \in R$ such that $eR = fR$ and $(\overline{e} - er\overline{e})R = \overline{f}R$. Clearly $f = e + er\overline{e}$ suits well for this. ■

Remark 7 *Actually $R(\overline{e} - er\overline{e}) = R\overline{e}$.*

Further, notice another easy exercise

Lemma 8 *Let A, B be direct summands of a module M_R and $A \cap B = \{0\}$. The following statements are equivalent*
(a) A has a direct complement which includes B;
(b) $A \oplus B$ is a direct summand in M.

If any of these conditions holds, we say M *has enough direct complements* (see [6]).

Finally,

Proposition 9 *If R is a regular ring then R_R has enough direct complements.*

Proof. Suppose $eR \cap fR = \{0\}$. According to our Theorem, since $(f - fx\overline{e}f)R = eR \cap fR$, $f - fx\overline{e}f = 0$ follows (here $\overline{e}fx\overline{e}f = \overline{e}f$ by regularity). Hence $(\overline{e} + efx\overline{e})f = (\overline{e} + e)f = f$ and so, choosing $r = -fx$ (in Lemma 6) and $g = (\overline{e} - er\overline{e})f$, we obtain $f \in gR$ that is, $fR \subseteq gR$ and $eR \oplus gR = R$. ■

7 Comments

While von Neumann's proof obviously can be also used in order to show that the existence of the inf's implies the existence of the sup's, dualization for Lam's proof is not straightforward.

Indeed, automatic use of 9's Lemma (everything needed is on the following diagram with exact rows and columns) does not work (in either direction):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A\cap B & \longrightarrow & A & \longrightarrow & \frac{A+B}{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & R & \longrightarrow & R/B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{A+B}{A} & \longrightarrow & R/A & \longrightarrow & \frac{R}{A+B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array},$$

that is, splitting 2-nd and 3-rd exact rows do not generally imply splitting 1-st row (nor dually, splitting 1-st and 2-nd exact rows do not imply splitting 3-rd row). One has to involve projective modules.

For intersection: since B is direct summand in R, R/B is projective. Since $A+B$ is direct summand in R, so is $\frac{A+B}{B}$ in R/B. Hence $\frac{A+B}{B}$ is projective, $A \longrightarrow \frac{A+B}{B}$ (co)splits and so $A\cap B \longrightarrow A$ splits. Thus $A\cap B$ is direct summand in A and so, in R.

For sum: the pushout

$$\begin{array}{ccc} R & \longrightarrow & R/B \\ \downarrow & & \downarrow \\ R/A & \longrightarrow & \frac{R}{A+B} \end{array}$$

with projective R/A, R/B should be used. But a pushout of projective modules is not generally projective (this way $A\cap B$ would not be involved).

Therefore something different is needed. Not homological but revealing something related to regular rings, here is

Another solution.

Theorem 10 *Let A, B and $A\cap B$ be direct summands in R_R. If R is a regular ring then $A+B$ is a direct summand of R_R, too.*

Proof. By modularity, $A\cap B$ is direct summand in A, say $A=(A\cap B)\oplus C$, and in B, say, $B=(A\cap B)\oplus D$. Simple computation shows that $A+B=B\oplus C$.

Further, since B is a direct summand in R_R and $B\cap C=\{0\}$, $R=B\oplus E$ where E can be chosen (see Application above) with $C\leq \dot{E}$. Since C is a direct summand of R_R, it also direct summand of E, say $E=C\oplus F$. Finally, $R=B\oplus E=B\oplus C\oplus U=(A+B)\oplus U$, as desired. ∎

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