Von Neumann Regular 2×2 matrices over integral domains

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For a ring R, denote by $reg(R)$ the (von Neumann) regular elements, that is, the elements $a \in R$ for which an $x \in R$ exists such that $a = axa$. Sometimes, x is called an inner inverse for a.

In this short note we present *an undergraduate approach* to regular matrices of size 2 and (partly) 3.

Simple remarks. 1) If $a \in reg(R)$ in any (unital) ring R and $u \in U(R)$ then both $au, ua \in reg(R)$ [incl. $-a$].

[Proof for $au = axau = (au)u^{-1}x(au)$]. Hence, (unit-)regularity is *invariant to association.*

2) Obviously 0 and the units [$x = a^{-1}$] are (unit-)regular in any ring.

3) Suppose R is an integral (commutative) domain and let $S := \mathbb{M}_n(R)$. If $A = AXA$, taking determinants, $\det(A)(\det(AX) - 1) = 0$ so $\det(A) = 0$ or else $\det(AX) = 1$ (and also $\det(XA) = 1$). Hence AX and XA are units, and since the matrix ring is Dedekind finite, both A, X are units.

4) Suppose d is the gcd (if any) of the entries of a regular matrix A. Then d is an idempotent.

Therefore only the $\det(A) = 0$ case remains to be settled.

1 2×2 matrices

Lemma 1 *If* $\mathrm{char}(R) \neq 2$, *for a* 2×2 *matrix over any commutative ring,* $\det(A) = 0$ *iff* $\mathrm{Tr}(A^2) = \mathrm{Tr}^2(A)$.

Proof. If $\det(A) = 0$ by Cayley-Hamilton' theorem, $A^2 = \mathrm{Tr}(A)A$. Taking traces gives $\mathrm{Tr}(A^2) = \mathrm{Tr}^2(A)$. Conversely, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the condition yields $a^2 + 2bc + d^2 = (a + d)^2$ which gives $ad = bc$, i.e. $\det(A) = 0$. $\blacksquare$

In the sequel, we say elements a, b, c, d are coprime (or, equivalently, the row $\begin{bmatrix} a & b & c & d \end{bmatrix}$ is unimodular) if there exist elements x, y, z, t such that $ax + cy + bz + dt = 1$.

In the $n = 2$ case it is easy to prove the following characterization

Theorem 2 *Let R be a commutative domain. A nonzero 2×2 matrix with zero determinant is (von Neumann) regular iff its nonzero entries are coprime.*

Proof. Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0_2$ with $ad = bc$ and $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AXA = A$ amounts to a (nonhomogeneous) system, namely

$$\begin{array}{c} a^2x + acy + abz + bct = a \\ abx + ady + b^2z + bdt = b \\ acx + c^2y + adz + cdt = c \\ bcx + cdy + bdz + d^2 = d \end{array}.$$

Since $ad = bc$, the system reduces to

$$\begin{array}{c} a(ax + cy + bz + dt) = a \\ b(ax + cy + bz + dt) = b \\ c(ax + cy + bz + dt) = c \\ d(ax + cy + bz + dt) = d \end{array}.$$

If any of a, b, c, d is zero, the corresponding equality holds for any x, y, z, t.

Since we have assumed $A \neq 0_2$, at least one entry (say a) is nonzero. Dividing by a the first equation, we get $ax + cy + bz + dt = 1$, which holds iff a, b, c, d are coprime. ■

Remarks. 1) Notice that the *domain* hypothesis is used just for the necessity.

2) In the above statement, if three entries are zero, the fourth must be a unit, i.e. the matrix is of form uE_{11}, uE_{12}, uE_{21}, uE_{22} with $u \in U(R)$. If $R = \mathbb{Z}$, the fourth must be $= \pm 1$, i.e., we get the matrices $\pm E_{11}$, $\pm E_{12}$, $\pm E_{21}$, $\pm E_{22}$.

Summarizing, *regular* 2×2 integral matrices are (zero and the) units, which are unit-regular and so regular, and, rank 1 matrices with coprime nonzero entries. These are $\pm E_{11}$, $\pm E_{12}$, $\pm E_{21}$, $\pm E_{22}$, the matrices with two nonzero coprime entries, and the matrices with four nonzero (collectively) coprime entries [only one zero, not possible].

3) The system obtained in the previous proof also gives an inner inverse for any regular 2×2 matrix. We just have to chose x, y, z, t in $ax+cy+bz+dt = 1$, corresponding to the nonzero coprime entries, and zero for the zero ones. See examples below.

4) Using the above characterization, it is easy to give an example which shows that $reg(R)$ is *not multiplicatively closed*:

take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ which have nonzero coprime entries (both idempotents); then $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is not regular in any ring R with $2 \notin U(R)$. Moreover, $BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is regular by the theorem. Indeed, $X = E_{11}$ is an inner inverse [using the $ax + cy + bz + dt = 1$: $a = x = 1$, all the others, zero].

Question. If $\gcd(a;b;c;d)=1$ and $ad=bc$, does it follow that at least two are coprime ?

Examples. 1) $A=\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$; take $x=4$, $z=-1$, $y=t=0$. One can check $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$.

2) Notice that three nonzero entries, and only one zero, contradicts $ad=bc$. So the possible regular matrices with zero determinant are uE_{11}, uE_{12}, uE_{21}, uE_{22} with $u\in U(R)$, with three zeros, two zeros and two coprime entries or else all four nonzero (collectively) coprime entries with $ad=bc$.

$A=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; take $x=5$, $t=-1$, $y=z=0$.

One can check $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Observe that this happens for all zero determinant matrices which have at least one entry $=1$, or, at least two coprime entries.

3) $A=\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$; take $x=2$, $z=-1$, $y=t=0$. One can check $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$.

$$1=ax+cy+bz+dt=\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}+\begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} y \\ t \end{bmatrix}.$$

2 3×3 matrices

Is this true for 3×3 matrices ? **NO**.

Partial check for 3×3. We write just the first equality (out of 9).

Denote $A=[a_{ij}]$, $X=[x_{ij}]$, $1\leq i,j\leq 3$.

$\text{row}_1(AX)=[a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31},\ a_{11}x_{12}+a_{12}x_{22}+a_{13}x_{32},\ a_{11}x_{13}+a_{12}x_{23}+a_{13}x_{33}]$ and

$\text{col}_1(A)=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$ yield the first equation:

$S_{11}=a_{11}(a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31})+a_{21}(a_{11}x_{12}+a_{12}x_{22}+a_{13}x_{32})+a_{31}(a_{11}x_{13}+a_{12}x_{23}+a_{13}x_{33})=a_{11}$.

If $\det(A)=0$ can we factor out a_{11} ? Moreover, do we get

$$a_{11}(\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}+\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}+$$
$$+\begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix})=a_{11}\ ?$$

If so, we have again the coprime condition.

From the above sum S_{11} we already have terms which have the factor a_{11}, i.e.

$a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{21}x_{12} + a_{31}x_{13}$,

and another four terms, i.e. $a_{21}(a_{12}x_{22} + a_{13}x_{32}) + a_{31}(a_{12}x_{23} + a_{13}x_{33})$.

Can we express these with the factor a_{11} because of $\det(A) = 0$?

The remaining terms should be

$a_{11}\left(\begin{bmatrix} a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_{22} \\ x_{32} \end{bmatrix} + \begin{bmatrix} a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{23} \\ x_{33} \end{bmatrix}\right)$ which amounts to

$a_{11}a_{22} = a_{12}a_{21}$, $a_{11}a_{23} = a_{13}a_{21}$, $a_{11}a_{32} = a_{12}a_{31}$, $a_{11}a_{33} = a_{13}a_{31}$.

That this, the vanishing of the cofactors in A which include a_{11}:

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = 0.$

For a_{12} this reduces to the vanishing of the cofactors which include a_{12}, and so on.

Finally, for an analogous characterization, we need all 2×2 cofactors to be zero, i.e. $\mathrm{rank}(A) = 1$.

Which is clearly stronger than $\det(A) = 0$.

We have (partly) obtained the following

Proposition 3 *Let R be a commutative domain. A nonzero 3×3 matrix of rank 1 is (von Neumann) regular iff its nonzero entries are coprime.*

Finally, here is an example of rank 2 regular 3×3 matrix with not coprime entries. As noticed in the simple remark (4), the gcd of the entries should be an idempotent.

Example. Consider $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3(E_{11} + E_{22}) \in \mathbb{M}_3(\mathbb{Z}_6)$. Clearly, $A = AXA$ for $X = E_{11} + E_{22}$ since $3^2 = 3$ over $\mathbb{Z}_6$ (which is a commutative ring, but not a domain).

3 Lam, Swan

In [1], the following development of the elementary results in the previous section is given.

For any matrix $A \in \mathbb{M}_n(R)$ (where R is a commutative ring), let $\mathcal{D}_i(A)$ $(1 \leq i \leq n)$ denote the i-th *determinantal ideal* of A, that is, the ideal in R generated by the $i \times i$ minors of A. We have a descending sequence $\mathcal{D}_0(A) \supseteq \mathcal{D}_1(A) \supseteq \cdots \supseteq \mathcal{D}_n(A) = \det(A) \cdot R \supseteq (0)$, where, by convention, $\mathcal{D}_0(A) = R$.

Theorem 4 *A matrix $A = (a_{ij}) \in \mathbb{M}_n(R)$ is von Neumann regular iff each determinantal ideal $\mathcal{D}_i(A)$ $(0 \leq i \leq n)$ is idempotent (or equivalently, each $\mathcal{D}_i(A)$ is generated by an idempotent in R).*

For the last equivalence, we use the well-known fact that *a f.g. ideal is idempotent iff it is generated by an idempotent.*

Remark. In the case where R is a *connected* ring [i.e. has only trivial idempotents], the theorem shows that A is von Neumann regular iff each $\mathcal{D}_i(A)$ is either (0) or R.

The small size special cases which appear are the following

Proposition 5 *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* $ad = bc$ *and* $\mathcal{D}_1(A) = eR$*, where* $e = e^2$*. Fix an equation* $aw + bx + cy + dz = e$*. Then the matrix* $M = \begin{bmatrix} w & y \\ x & z \end{bmatrix}$ *satisfies* $A = AMA$ *(so* A *is von Neumann regular, with quasi-inverse* M*).*

Proposition 6 *The (alternating) matrix* $A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ *is von Neumann regular iff* $aR + bR + cR = eR$ *for some idempotent* $e \in R$*. (In this case,* A *is in fact unit-regular.) In particular,* A *is von Neumann regular if the row* (a, b, c) *is unimodular; in case* R *is connected and* $A \neq 0$*, the converse holds.*

References

[1] T. Y. Lam, Swan *Symplectic modules and von Neumann regular matrices over commutative rings.* (2010) In: Van Huynh D., López-Permouth S.R. (eds) Advances in Ring Theory. Trends in Mathematics. Birkhäuser Basel, p. 213-227.