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A solution to a problem on lattice isomorphic Abelian groups

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Abstract. This paper completes the solution to the problem of deciding when two Abelian groups will have the lattices of their subgroups isomorphic.

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1 Introduction

This paper completes the last step in the investigation initiated by R. Baer [1] to answer the question: when two Abelian groups G and H have their subgroup lattices $L(G)$ and $L(H)$ isomorphic? In this case, following Baer [1], we say that G and H are *projective* groups. Note that if G is projective with H, then G need not be isomorphic to H as is clear when G and H are cyclic groups of order p^n and q^n respectively, where p and q are different prime numbers.

R. Baer made substantial progress in solving this problem and proved, among other things, the following major results:

Theorem 1.1 ([1], [2]). *Let G and H be two Abelian groups and $L(G)$ and $L(H)$ be their subgroup lattices.*

(a) *If G has torsion-free rank > 1, then $L(G) \cong L(H)$ if and only if $G \cong H$.*

(b) *If G is a torsion group, then $L(G) \cong L(H)$ if and only if there is a bijection between the primary components of G and H such that the corresponding primary components P and Q are isomorphic whenever* rank $P > 1$, *and if P has rank 1*, say *$P \cong \mathbf{Z}(p^n)$ for some prime p with $n > 0$ or $n = \infty$, then the corresponding primary component Q is isomorphic to $\mathbf{Z}(q^n)$ for some (perhaps different) prime q.*

(c) *If G is torsion-free and has rank 1, then $L(G) \cong L(H)$ implies that H is a torsion-free group of rank 1*. Moreover, *$G \cong H$, if G is, in addition, infinite cyclic*.

L. Fuchs([2]) extended Theorem 1.1 (c) by proving the following:

Theorem 1.2 ([3]). *Let G be a rank 1 torsion-free Abelian group of type*

$$(k_1, ..., k_n, ...).$$

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If G is projective with an Abelian group H, then H is a rank 1 torsion-free group with type $(l_1, ..., l_n, ...)$ where the l_i's are obtained from the k_i's by a permutation π of the indexing set of primes.

Extracting some of the ideas from the paper by R. Baer ([1]) and L. Fuchs [2] and making corrections, K. Mahdavi and J. Poland ([5]) obtained the following necessary condition for two mixed groups of torsion-free rank 1 to be projective:

Theorem 1.3 ([5]). *Let G and H be two mixed Abelian groups of torsion-free rank 1. If $f : L(G) \to L(H)$ is a lattice isomorphism, then the torsion part $T(G) \cong T(H)$, the height matrix $U(H)$ results from $U(G)$ by a permutation π of the primes (the rows of $U(G)$) which fixes the primes p occurring as orders of elements in G and π and f are related by the property that for any subgroup S of G and prime p, $f(pS) = \pi(p)f(S)$.*

The permutation π in Theorem 1.3 is actually *multiplicative*, that is, it extends to an endomorphism of the multiplicative semigroup of positive integers. Further, as pointed out in ([5]), the equation $f(nS) = \pi(n)f(S)$ holds for all positive integers n and for all subgroups S of G.

An example of Megibben (see [7], [6]) shows that the converse of Theorem 1.3 is false. This raises the problem of describing the mixed Abelian groups G and H with torsion-free rank 1 for which $L(G) \cong L(H)$.

K. Mahdavi and J. Poland ([5], [6]), and independently, U. Ostendorf ([8]), partially answered this by considering the special case when G is a splitting mixed Abelian group.

Our main theorem answers this problem completely.

Theorem 1.4. *Let G and H be mixed Abelian groups of torsion-free rank 1. Then $L(G) \cong L(H)$ if and only if*

(i) *G and H have isomorphic torsion parts: $T(G) \cong T(H)$;*

(ii) *There exist elements of infinite order $a \in G$, $b \in H$ such that the height matrix $\mathcal{H}(b) = U(H)$ arises from $\mathcal{H}(a) = U(G)$ by a permutation π of primes which fixes those primes occurring as orders of elements in G and is multiplicative;*

(iii) *There is a bijection between the p-components of $G/\langle a\rangle$ and $H/\langle b\rangle$ such that the corresponding components are isomorphic if they have rank >1 and, if for a prime p, $(G/\langle a\rangle)_p$ has rank 1 and corresponds to $(H/\langle b\rangle)_q$ for some (not necessarily distinct) prime q, then they are of the form $\mathbf{Z}(p^n)$ and $\mathbf{Z}(q^n)$ respectively, where n is a positive integer or ∞.*

Observe that condition (iii) simply states that $G/\langle a\rangle$ and $H/\langle b\rangle$ are projective.

Our methods involve using a key idea of constructing a projectivity of a divisible group by Mahdavi and Poland [5] and, following Warfield, viewing a mixed Abelian group as an extension of a torsion-free group by a torsion group.

Thus Theorem 1.4 together with Theorem 1.1 provides a complete solution to the problem of describing when two Abelian groups have isomorphic subgroup lattices.

2 Preliminaries

Let $\mathbf{P}$ be the set of all prime integers. All groups that we consider are additively written Abelian groups and we generally follow the notation and terminology of the books by L. Fuchs [3] and R. Schmidt [9]. For an Abelian group G, $L(G)$ denotes the lattice of subgroups of G under set inclusion. $T(G)$ denotes the torsion part of G . For a prime p, G_p denotes the p-component of $T(G)$ and $G[p]$ denotes the subgroup $\{a \in G \mid pa = 0\}$. For a subset S of G, $\langle S \rangle$ denotes the subgroup generated by S. If $g \in G$, $\langle g \rangle$ denotes $\langle \{g\} \rangle$, the cyclic subgroup generated by g. For all ordinals α, the subgroups $p^\alpha G$ are defined inductively: $pG = \{pa \mid a \in G\}$; if $\alpha = \gamma + 1$ then $p^{\gamma+1}G = p(p^\gamma G))$ and if α is a limit ordinal, then $p^\alpha G = \bigcap\limits_{\gamma < \alpha} p^\gamma G$.

If $a \in p^\alpha G$ and $a \notin p^{\alpha+1}G$, we say a has *p-height* α and write $h_p(a) = \alpha$. If $a \in p^\alpha G$ for all α, then a has infinite height and we write $h_p(a) = \infty$.

The *height matrix* of an element a in a group G is a doubly infinite matrix $\mathcal{H}(a)$ indexed by the primes p and non-negative integers n whose (p, n) entry is $h_p(p^n a)$. If G is a mixed group of torsion-free rank one and if a, b are two infinite order elements in G, then $\mathcal{H}(a)$ and $\mathcal{H}(b)$ are *equivalent*, that is, almost all rows are equal and if the p^{th}-rows of $\mathcal{H}(a)$ and $\mathcal{H}(b)$ are not equal, then there exist integers n and m such that $h_p(p^{n+i}a) = h_p(p^{m+i}b)$ for every $i \in \{0, 1, 2, ...\}$. Thus to each mixed Abelian group G of torsion-free rank one we can assign uniquely an equivalence class $U(G)$ of the height matrix of any infinite order element in G. The p^{th}-row of $\mathcal{H}(a)$ is called the *p-indicator* of a.

An indicator $(\sigma_0, \sigma_1, ...)$ is said to have a *gap* if, for some $k \geq 0$, $\sigma_k + 1 < \sigma_{k+1}$; in this case the gap is said to *follow* σ_k or *precede* σ_{k+1}. For any prime p, the α^{th} Ulm-Kaplansky invariant of G is denoted by $f^p_\alpha(G)$ and is defined as the dimension $\dim\left(\frac{p^\alpha G[p]}{p^{\alpha+1}G[p]}\right)$. We shall be using the well-known result (see [3]), that two countable Abelian p-groups with the same Ulm-Kaplansky invariants are isomorphic.

3 The main result

We begin by mentioning two lemmas by R. Hunter [4] about mixed groups of torsion-free rank 1.

Lemma 3.1 ([4]). *Let G be a mixed Abelian group of torsion-free rank one and $a \in G$ an element of infinite order and p-indicator $(\sigma_0, \sigma_1, ...)$. Then, for any ordinal σ,*

$$f^p_\sigma(G) = \begin{cases} f^p_\sigma(G/\langle a \rangle) + 1 & \textit{if } \sigma = \sigma_n \textit{ and a gap follows } \sigma_n \\ f^p_\sigma(G/\langle a \rangle) - 1 & \textit{if } \sigma + 1 = \sigma_n \textit{ and a gap precedes } \sigma_n \\ f^p_\sigma(G/\langle a \rangle) & \textit{otherwise} \end{cases} .$$

Lemma 3.2 ([4]). *If G and H are mixed Abelian groups of torsion-free rank 1, then $G \cong H$ if and only if there exist infinite order elements $a \in G$, $b \in H$ such that $\mathcal{H}(a) = \mathcal{H}(b)$ and $G/\langle a \rangle \cong H/\langle b \rangle$.*

The next result (which is Theorem 1.3.2 of [9]) states that certain bijections between cyclic subgroups induce projectivities.

Proposition 3.3 ([9]). *Let G and H be two Abelian groups. If g is a bijection from the set of cyclic subgroups of G to the set of cyclic subgroups of H, then g extends to a projectivity f from G to H if g satisfies the property*

$$A \leq B + C \Leftrightarrow g(A) \leq g(B) + g(C) \tag{3.1}$$

for all cyclic subgroups A, B, C of G.

We now consider the construction of autoprojectivities of a divisible Abelian group of torsion-free rank one corresponding to specific permutations of the set $\mathbf{P}$ of primes. The following result is a special case of Theorem E of [5]. For the sake of completeness, we shall give a (slightly simpler) proof which is more group-theoretical.

Following L. Fuchs [3], a prime p is said to be *relevant* to a group G, if G has an element of order p. A positive integer n is said to be *relevant* to a group G if each prime factor of n is relevant to G.

Proposition 3.4 ([5]). *Let D be a divisible mixed Abelian group of torsion-free rank one and let π be a permutation of the set $\mathbf{P}$ of primes which is multiplicative and fixes those primes that are relevant to D. Then there is a lattice isomorphism $f : L(D) \longrightarrow L(D)$ such that $f(nS) = \pi(n)f(S)$ for all subgroups S of D and all positive integers n.*

Proof. Let $D = T \oplus \mathbf{Q}$ with T torsion divisible and $\mathbf{Q}$ the additive group of rational numbers. Let P' be the set of primes relevant to T. Let K be the localization of $\mathbf{Z}$ at the primes not relevant to T, that is, K is the subgroup of $\mathbf{Q}$ generated by $\{\frac{1}{p^k} : p \in P', k \in \mathbf{Z}\}$ i.e., $K = \langle \{1/m : m \text{ positive integer relevant to } T\} \rangle$, so that $\mathbf{Z} < K$ and $K/\mathbf{Z} \cong \bigoplus \{\mathbf{Z}(p^\infty) : p \in P'\}$. Clearly K is the union of cyclic subgroups $C_m = \langle x_m \rangle$, where $x_m = 1/m$ for various positive integers m relevant to T. Note that if $m = rs$, then $rx_m = x_s$ and x_m is in $\langle x_n \rangle$ if and only if $m|n$.

Let $D = T \oplus Q'$ with $Q' \cong \mathbf{Q}$ be another decomposition of D with $\langle y \rangle = Z' < Q'$ and $Z' \cong \mathbf{Z}$. Choose $K' > Z'$ analogous to the subgroup K , so that $K'/Z' \cong \bigoplus \{\mathbf{Z}(p^\infty) : p \in P'\}$ and we realize $Q'/K' \cong \{\mathbf{Z}(p^\infty) : p \in \pi(P) \backslash P'\}$. As before, K' is the union of cyclic subgroups $C'_m = \langle y_m \rangle = \langle 1/m \rangle$ for various positive integers m relevant to T with y_m satisfying $my_m = y$ and if $m = rs$, then $ry_m = y_s$.

Let $H = T \oplus K$ and $H' = T \oplus K'$. If $a \in D$ is an element of infinite order with $o(a + H) = e$, then there is a smallest integer n such that $ea \in T \oplus C_n$ so that

$$ea = u + dx_n = d(v + x_n) \text{ with } u = dv \in dT = T. \tag{3.2}$$

Similar equation (3.2) holds with respect to $T + H'$.

Now given such an a satisfying (3.2), choose b in D such that

$$\pi(e)b = \pi(d)(v + y_n). \tag{3.3}$$

Such an element b exists and is unique since D is divisible and $\pi(e)$ is not relevant to $T \oplus H'$. Note that if a is not in H, then b is not in H' as $\pi(e)$ is not relevant to $H'/(T \oplus Z')$. Also the minimality of e and n implies that gcd$(d, e) = 1 =$ gcd(d, n).

We now wish to define a bijection f from the set of cyclic subgroups of D to itself satisfying the property (3.1) of Proposition 3.3.

To this end, we use the natural isomorphism $K \longrightarrow K'$ and the identity map $T \longrightarrow T$, and define the function g by

$$g(\langle a\rangle) = \begin{cases} \langle a\rangle & \text{if } a \in T \\ \langle ry_n\rangle & \text{if } a = rx_n \in H \\ \langle b\rangle & \text{if } a \text{ has infinite order satisfying (3.2), and } b \text{ satisfies (3.3)} \end{cases}.$$

Using the uniqueness of e, d, m and the inverse permutation π^{-1}, it is easy to see that g is a bijection between cyclic subgroups of $T \oplus \mathbf{Q}$ and $T \oplus Q'$ respectively.

To finish off the proof, we need only verify the condition (3.1) of Proposition 3.3 for the non-trivial case when $\langle z\rangle < \langle t\rangle \oplus \langle w\rangle$, where $w \in T$ and z, t have infinite order and then apply Lemma 3.1 of [5]. Specifically, if $e_1 z = d_1(v_1 + x_m)$ and $e_2 t = d_2(v_2 + x_n)$ as per equation (3.2) and $z = at + bw$ with a, b in $\mathbf{Z}$, then a direct calculation as done in the proof of Lemma 3.1 of [5] shows that m, e_1, d_2 are divisors respectively of n, e_2 and d_1 so that $n = a_1 m$, $e_2 = a_2 e_1$, and $d_1 = a_3 d_2$, $a = a_1 a_2 a_3$ and $d_1(v_1 - a_1 v_2) = e_1 bw$ and conversely. Since π is multiplicative, the corresponding elements z',t' where $g(\langle z\rangle) = \langle z'\rangle$ and $g(\langle t\rangle) = \langle t'\rangle$, satisfy the similar equation needed to reach the conclusion that $\langle z'\rangle$ is in $\langle t'\rangle \oplus \langle w\rangle$ and conversely. Hence the condition (3.1) of Proposition 3.3 holds and so g induces a projectivity f from D to D that satisfies $f(nS) = \pi(n)f(S)$ for all cyclic (and hence any) subgroup S of D and any positive integer n. $\square$

Proof of Theorem 1.4

The conditions are necessary: Suppose $f : L(G) \to L(H)$ is a lattice isomorphism and let $a \in G$ be an infinite order element with $\mathcal{H}(a) = U(G)$. If for some $b \in H$, $f(\langle a\rangle) = \langle b\rangle$, then by Theorem 1.3, conditions (i) and (ii) are satisfied. Moreover, f induces a lattice isomorphism $\overline{f} : L(G/\langle a\rangle) \longrightarrow L(H/\langle b\rangle)$. Then, by Theorem 1.1(b), condition (iii) holds.

The conditions are sufficient: Suppose G and H satisfy conditions (i)-(iii). Identifying the divisible hulls of $T(G)$ and $T(H)$, we may assume, without loss of generality, that G and H are (essential) subgroups of $D \oplus \mathbf{Q}$, where D is a torsion divisible Abelian group and $\mathbf{Q}$ is the additive group of rational numbers.

Now apply Proposition 3.4 to construct a projectivity $f : D \oplus \mathbf{Q} \longrightarrow D \oplus \mathbf{Q}$ corresponding to the permutation π with the property that for all subgroups S of $D \oplus \mathbf{Q}$ and primes p, $f(pS) = \pi(p)(f(S))$.

So if $K = f(G)$ and $\langle c\rangle = f(\langle a\rangle)$, then G is projective with K, $\mathcal{H}(c)$ arises from $\mathcal{H}(a)$ by permuting the rows of $\mathcal{H}(a)$ by π and $G/\langle a\rangle$ is projective with $K/\langle c\rangle$. By Theorem 1.3, $T(K) \cong T(G)$. In view of condition (iii) of our hypothesis, Theorem 1.1(b), shows that $G/\langle a\rangle$ is projective with $H/\langle b\rangle$. Putting these facts together, we conclude that $T(K) \cong T(H)$, $\mathcal{H}^K(c) = \mathcal{H}^H(b)$ and $K/\langle c\rangle$ is projective with $H/\langle b\rangle$. By Theorem 1.1(b), there is a bijection between the p-components of the torsion groups $K/\langle c\rangle$ and $H/\langle b\rangle$ such that the corresponding components are projective and those of rank >1 are isomorphic.

Now $T(K) \cong T(H)$ implies the equality of the Ulm-Kaplansky invariants $f^p_\sigma(K) = f^p_\sigma(H)$ for all primes p and ordinals σ. Since $\mathcal{H}^K(c) = \mathcal{H}^H(b)$, Lemma 3.1 implies that $K/\langle c\rangle$ and $H/\langle b\rangle$ have the same Ulm-Kaplansky invariants, that is $f^p_\sigma(K/\langle c\rangle) = f^p_\sigma(H/\langle b\rangle)$.

Let $P' = \{p \text{ prime}|\text{rank}(K/\langle c\rangle)_p = 1\}$. If, for some $p \in P'$, $(K/\langle c\rangle)_p \cong \mathbf{Z}(p^n)$ with n a positive integer or ∞, then from the equality of the Ulm-Kaplansky invariants $f^p_\sigma((K/\langle c\rangle)_p) = f^p_\sigma(K/\langle c\rangle) = f^p_\sigma(H/\langle b\rangle)$, which holds for all ordinals σ, we derive $(H/\langle b\rangle)_p \neq 0$ and cannot have rank > 1. Hence $(H/\langle b\rangle)_p$ has rank 1 and is therefore a countable p -group with the same Ulm-Kaplansky invariants as $(K/\langle c\rangle)_p$ and by [3], $(K/\langle c\rangle)_p \cong (H/\langle b\rangle)_p$. Thus $K/\langle c\rangle \cong H/\langle b\rangle$. Finally, since $\mathcal{H}^K(c) = \mathcal{H}^H(b)$, we conclude, by Lemma 3.2, that $K \cong H$. Hence G is projective with H and the proof is complete.

4 Corollaries

A *simply presented* group is an Abelian group G defined by generators and defining relations such that all the relations are induced by relations involving two generators, having the form $px = y$ or $px = 0$ where x, y belong to the given generating set and p varies over primes. A direct summand of a simply presented group is called a *Warfield* group.

First we specialize Theorem 1.4 to Warfield groups.

Corollary 1. *If G and H are Warfield groups of torsion-free rank one, then $L(G) \cong L(H)$ if and only if G and H satisfy conditions (i) and (ii) of Theorem 1.4.*

Proof. All one needs is to observe that in the proof of Theorem 1.4, $K/\langle c\rangle$ and $H/\langle b\rangle$ are now simply presented torsion groups with the same Ulm-Kaplansky invariants and hence are isomorphic. ∎

Corollary 2. *Suppose G and H are mixed Abelian groups of torsion-free rank one and, for every prime p, G has an element of order p. Then $L(G) \cong L(H)$ if and only if $G \cong H$.*

Proof. Let $f : L(G) \to L(H)$ be the lattice isomorphism. By Theorem 1.4 (i), (ii), $T(G) \cong T(H)$ and, since π now fixes every prime p, there exist $a \in G$, $b \in H$ such that

$\mathcal{H}(a) = U(G) = U(H) = \mathcal{H}(b)$, where $f(\langle a \rangle) = \langle b \rangle$. Since $T(G) \cong T(H)$, Lemma 3.1 implies that the torsion groups $G/\langle a \rangle$ and $H/\langle b \rangle$ have the same Ulm-Kaplansky invariants.

Now proceed as in the proof of Theorem 1.4 replacing the role of K by G, to conclude $G/\langle a \rangle \cong H/\langle b \rangle$. Then Lemma 3.2 yields that $G \cong H$. $\square$

Corollary 3. *Suppose G and H are mixed Abelian groups of torsion-free rank one and $T(G)$ is a p-group for some prime p. Then $L(G) \cong L(H)$ if and only if conditions (i) and (ii) of Theorem 1.4 hold and $G/\langle a \rangle \cong H/\langle b \rangle$, whenever $T(G)$ has rank > 1.* $\square$

References

[1] Baer R., *The significance of the system of subgroups for the structure of a group*, Amer. Journ. Math., 71 (1939), 1 - 44.

[2] Fuchs L., *Abelian Groups*, Publishing House of Hungarian Academy of Sci., Budapest (1958)

[3] Fuchs L., *Infinite Abelian Groups*, Academic Press, vol. 1 & 2, 1970, 1973.

[4] Hunter R., *Balanced subgroups of Abelian groups,* Trans. Amer. Math. Soc., 215 (1976), 81 - 98.

[5] Mahdavi K., Poland J., *On lattice-isomorphic abelian groups.* Arch. Math. (Basel) 58 (1992), no. 3, 220 – 230.

[6] Mahdavi K., Poland J., *On lattice isomorphic of mixed abelian groups.* Arch. Math. (Basel) 60 (1993), no. 4, 327 – 329.

[7] Megibben C. K., *On mixed groups of torsion-free rank 1*, Illinois J. of Math., 11, (1967), 134 - 144.

[8] Ostendorf U., *Projektivitätstypen torsionsfreier abelscher Gruppen vom Rang 1.* (German) [Projectivity types of torsion-free abelian groups of rank 1] Rend. Sem. Mat. Univ. Padova 86 (1991), 183 – 191.

[9] Schmidt R., *Subgroup lattices of groups*, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, 1994.

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