SQUARE ROOTS OF QUASIREGULAR ELEMENTS IN A RING, ARE QUASIREGULAR

First recall some well known exercises and definition (e.g., [2], **4.1**, **4.2**) which are stated for R, a ring possibly without 1.

E1. Define $a \circ b = a + b - ab$. Show that this binary operation is associative, and $(R, \circ)$ is a monoid with zero as the identity element.

D1. An element $a \in R$ is called *left* (or *right*) *quasi-regular* if a has a left (resp. right) inverse in the monoid $(R, \circ)$.

E2. Show that, if R has an identity, the map $\phi : (R, \circ) \to (R, \times)$, defined by $\phi(a) = 1 - a$ is a monoid isomorphism.

In this case, an element a is left (right) quasi-regular iff $1 - a$ has a left (resp. right) inverse with respect to ring multiplication.

In the sequel we give two solutions for

Exercise 1. *If a^2 is left (or right) quasi-regular, so is a.*

Adapted from [1], we first give the easy

solution in a ring possibly without 1.

Suppose a^2 is (say) right quasi-regular. There is $b \in R$ such that $a^2 \circ b = 0$. Notice that $a^2 = a \circ (-a)$. Hence

$$a \circ ((-a) \circ b) \overset{\text{assoc}}{=} (a \circ (-a)) \circ b = a^2 \circ b = 0$$

so $(-a) \circ b$ is a right inverse for a in the monoid $(R, \circ)$.

If R has identity, and we use the characterization, a is right quasi-regular iff $1-a$ has a right inverse,

Solution in a ring with identity.

Suppose a^2 is right quasi-regular, i.e., there exists $b \in R$ such that $(1 - a^2)b = 1$. Then $(1 - a)(1 + a)b = 1$ so $1 - a$ has a right inverse too.

Hence a is right quasi-regular.

Remark. Of course, the solutions correspond one another by the monoid isomorphism ϕ.

References

[1] I. Kaplansky *Fields and rings*. University of Chicago Press; 1st edition (1969), 198 p.

[2] T. Y. Lam *Exercises in classical ring theory*. Problem Books in Math. Springer (1995).