ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLV, s.I a, Matematică, 1999, f.1.

PURITY IN IDEAL LATTICES

BY

GRIGORE CĂLUGĂREANU

Abstract. In [4] T. HEAD gave a general definition of purity in complete lattices. In this note the pure elements of the lattice of all the nonempty ideals of a lattice are investigated.

1. Introduction. The utility of the purity as a property weaker then the direct summand property in commutative algebra is well–known.

In [4] an element p of a complete lattice L is called *pure* if for each compact element $k \in 1/p$, p has a complement in $k/0$. In a compactly generated lattice pure elements are more easily characterized.

The compact elements in a quotient sublattice b/a of a compactly generated lattice L are the elements $a \vee c$ with c compact in L and $a \vee c \leq b$. Hence, if L is compactly generated, each compact element in a quotient sublattice $1/p$ has the form $p \vee c$, with c a compact element in L.

Remark 1.1. [4] An element p is pure in a compactly generated lattice L if and only if for every element c compact in L, there is an element $d \in L$ such that $p \vee d = p \vee c$ and $p \wedge d = 0$.

In what follows we study the pure elements in the lattice $I(L)$ of all the nonempty ideals of a lattice L. Recall (see [2], [3] and [1]) that this is a (complete) compactly generated lattice, the compact elements being the principal ideals.

If $I, J \in I(L)$ we say that J is a *complement* of I in L (or, equivalently, that I, J are *complementary*), if $I \vee J = (I \cup J] = L$ (see [3]) and $I \wedge J = I \cap J = \{0\}$.

2. Elementary remarks.

Lemma 2.1. [4] *In a modular lattice each complemented element is pure.* ■

In the absence of the modularity, purity is no longer a weaker property than complementariness (e.g., in the 5-element non–modular (but complemented) lattice $\{0, a, b, c, 1\}$, $a < b$, the element a is complemented but not pure).

Moreover, in a complete lattice L each element is compact if and only if L is noetherian (i.e., has the ACC condition), so that *in a complete noetherian* (in particular finite or compact) *lattice, purity* is a *stronger* property than the *complementariness* (indeed, p is pure in L if and only if p has complement in each $c/0$, $p \leq c$, includes $c = 1$, with $1/0 = L$).

Combining these two facts we get

Theorem 2.1. *In a complete modular noetherian lattice an element is pure if and only if it has a complement.* ■

Further, each nonempty ideal in a lattice L is principal if and only if L is artinian (i.e., has the DCC condition) and each lattice L can be embedded by $a \mapsto a/0$ in its lattice of ideals $I(L)$, so that using

Consequence 2.1. *An artinian lattice is isomorphic with its ideal lattice*

and

Consequence 2.2. *In a lattice of finite length each element is compact and each ideal is principal*

one obtains

Theorem 2.2. *In a lattice of finite length an element p is pure if and only if p has a complement in each principal ideal that contains it.* ■

Theorem 2.3. *In a lattice of finite length an ideal is pure if and only if it is generated by a pure element.*

Proof. Using Consequence 2.2, a lattice of finite length is isomorphic with its lattice of ideals. So p has a complement in each $c/0$ (with $p \leq c$) if and only if $p/0$ has an ideal complement in each $(c/0)/\{0\}$ (with $p/0 \subseteq \subseteq (c/0)/\{0\}$, i.e., $p \leq c$). ■

A lattice is called *sectionally* (*or principally*) *complemented* if L has zero and all the sublattices $a/0$ ($a \in L$) are complemented, and *relative complemented* if each interval (quotient sublattice) is complemented. In a lattice with zero, $e \in L$ is called *essential* if $e \wedge a \neq 0$ for each $a \neq 0$.

Clearly, *each element in a sectionally* (*or relative*) *complemented lattice is pure.*

Moreover, a converse can be obtained using the following simple result

Lemma 2.2. *In a compactly generated lattice the only pure and essential element is* 1*.*

Proof. Let p be a pure and essential element in a compactly generated lattice L. According to Remark 1.1, for each compact element $c \in L$ there is an element $d \in L$ such that $p \vee c = p \vee d, p \wedge d = 0$. The element being essential, $d = 0$ and so $p = p \vee c$ or $c \leq p$. L being compactly generated $p = 1$. ∎

Then

Theorem 2.4. *In a compactly generated modular lattice each element is pure if and only if the lattice is sectionally complemented.*

Proof. Let us recall some auxiliary definitions and results:

In a lattice L with zero, b is called *a pseudocomplement* [see [5]] of a if b is maximal relative to the property $b \wedge a = 0$. L is called *pseudocomplemented* if each element has at least a pseudocomplement.

Each upper continuous [see [2]] (or, in particular, *compactly generated*) *lattice is pseudocomplemented.* Obviously each principal ideal $a/0$ of an upper continuous lattice is also an upper continuous sublattice, and hence it is pseudocomplemented.

If b *is a pseudocomplement of* a *in a modular lattice* L, *then* $a \vee b$ *is essential in* L.

Finally, if in a compactly generated (and hence upper continuous) modular lattice L each element is pure, for an arbitrary $a \in L$ let b be a pseudocomplement of c in $a/0$. Then $b \vee c$ is essential in $a/0$ and hence, using the previous Lemma, $b \vee c = 1$ so that b is also a complement of c in $a/0$. Notice that *in an upper continuous lattice, if* $p \leq a$ *and* p *is pure in* L, *then* p *is also pure in* $a/\dot{0}$. ∎

We insert here a

Consequence 2.3. *If* L *is modular, each ideal in* $I(L)$ *is pure if and only if* $I(L)$ *is sectionally complemented.*

Indeed, L is modular if and only if $I(L)$ is modular and $I(L)$ is compactly generated. ∎

Example. In a partition lattice each element (ideal) is pure.

Indeed, a partition lattice is relatively complemented so that each element is pure.

3. Pure ideals in bounded or complete lattices. First of all, $I(L)$ being a compactly generated lattice,

Remark 3.1. An ideal $P \in I(L)$ is pure if and only if for each principal ideal $a/0$ ($a \in L$), there is an ideal $I \in I(L)$ such that $P \cap I = \{0\}$ and $P \vee I = P \vee (a/0)$.

Hence, taking $a = 1$,

Remark 3.2. In a bounded lattice, each pure ideal P has a complement in $I(L)$. ■

Hence, *in a bounded lattice the purity in the lattice of all the ideals is stronger than complementariness.*

Moreover,

Theorem 3.1. *If L is a bounded modular lattice an ideal in $I(L)$ is pure if and only if it is complemented.*

Indeed, L is modular if and only if $I(L)$ is modular and one applies Lemma 2.1. ■

A little more can be said about the *principal* pure ideals and their connection with the complemented ones.

A simple case is the one of the artinian lattices. Using the previous remark and Consequence 2.1 we immediately obtain

Theorem 3.2. *Let L be an artinian lattice with a largest element and $p \in L$. If a (principal) ideal $p/0$ is pure then $p/0$ (or p) is complemented.*

Indeed, in this case L is bounded, all the ideals are principal, the lattice L is isomorphic with its ideal lattice and the ideal $p/0$ is complemented in $I(L)$ if and only if p is complemented in L. ■

Consequence 3.1. *Let L be an artinian modular lattice with a largest element and $p \in L$. Then $p/0$ is pure if and only if p is complemented.* ■

In a bounded lattice L, notice that $p/0$ is complemented in $I(L)$ if p is complemented in L but the converse does not generally hold.

Further

Theorem 3.3. *Let L be a bounded lattice. If $p \in L$ is a distributive complemented element in L then the principal ideal $p/0$ is pure in $I(L)$.*

Proof. If $p \vee q = 1, p \wedge q = 0$ then $(p/0) \vee ((q \wedge a)/0) = (p/0) \vee (a/0)$ and $(p/0) \wedge ((q \wedge a)/0) = \{0\}$.

Indeed, $p \vee (q \wedge a) \overset{distr}{=} (p \vee q) \wedge (p \vee a) = p \vee a$ and $p \wedge (q \wedge a) \leq p \wedge q = 0$ and the Remark 3.1 holds. ■

Consequence 3.2. *Let L be a bounded artinian lattice and let $p \in L$ be a distributive element. Then $p/0$ is pure if and only if p is complemented.* ■

From [3] recall that each standard element is a distributive element. Then

Theorem 3.4. *Let p be a standard element in a bounded lattice L. Then $p/0$ is pure if and only if p is complemented.*

Proof. Taking $a = 1$ in Remark 3.1, $p/0$ being pure in $I(L)$, there is an ideal $I \in I(L)$ such that $(p/0) \vee I = L = 1/0, (p/0) \wedge I = \{0\}$. These ideals being principal and p (or $p/0$) being standard, using Exercise 17, p.151 [3], the ideal I is also principal, say $I = q/0$. Hence q is a complement of p in L.

The converse follows using the above remark and theorem. ■

4. Pure ideals in noncomplete lattices with zero. Notice that we have not defined pure elements in a noncomplete lattice L with zero but even for such lattices pure ideals are defined $I(L)$ being (complete and) compactly generated. The example below was provided by Prof. G. Grätzer.

Set $\mathcal{N}(\mathbf{R}) = \{X \in \mathcal{P}(\mathbf{R}) | card(X) \leq \aleph_0\}$ and consider the corresponding lattice (together with the usual $\cap$ and $\cup$). Clearly $\emptyset$ is the smalest element and $\mathcal{N}(\mathbf{R})$ has no largest element. Hence this is not a bounded and so not a complete lattice. In fact, this is a noncomplete sublattice in the Boole algebra $(\mathcal{P}(\mathbf{R}), \subseteq)$ and so it is a distributive (and so also modular) lattice.

In $I(\mathcal{N}(\mathbf{R}))$ the principal ideal generated by X is $\mathcal{P}(X)$, the set of all the subsets of X.

Proposition 4.1. *Each principal ideal in $I(\mathcal{N}(\mathbf{R}))$ is complemented and hence pure.*

Indeed, each principal ideal $\mathcal{P}(X)$ has as complement $\mathcal{P}(\mathbf{R} - X) \cap \mathcal{N}(\mathbf{R})$ (recall that this lattice is distributive and $I \vee J = \{i \vee j | i \in I, j \in J\}$ holds, for $I, J \in I(L)$ in each distributive lattice L). ■

Remark 4.1. Not every ideal in this lattice is pure.

Indeed, consider in $\mathcal{N}(\mathbf{R})$ the set P of all the finite subsets of $\mathbf{R}$ is clearly an ideal (closed to unions and lower bounds).

P is an essential element in $I(\mathcal{N}(\mathbf{R}))$ (i.e., $\forall J \in I(\mathcal{N}(\mathbf{R})) : P \cap I \neq \{\emptyset\}$). Hence P has no (relative) complements and hence P is not pure nor complemented. ■

In this case (noncomplete lattice with zero) there seem to be no more general connections between pure elements and complements.

5. Pure–simple lattices of ideals. A complete lattice will be called *pure–simple* if its only pure elements are 0 and 1.

Some elementary results can be obtained using Lemma 2.2.

Consequence 5.1. *A compactly generated lattice such that all the nonzero elements are essential is pure–simple.* ∎

Consequence 5.2. *Let L be a lattice with zero. If all the nonzero ideals in $I(L)$ are essential then $I(L)$ is pure-simple.* ∎

Lemma 5.1. *If a lattice L has a smalest nonzero element then each element is essential. If L is finite or in L the meet of all the essential elements is nonzero the converse also holds.*

Proof. The first part is obvious, each meet of two nonzero elements containing the smalest nonzero element.

As for the converse, clearly L has at most one atom a (otherwise each atom would not be essential). This must be a lower nonzero element (for each $b \neq 0$, surely $a \wedge b \neq 0$ and then $a \wedge b = a$ or $a \leq b$). If L has no atoms both hypothesis imply that the meet u of all the essential elements must be nonzero. Moreover, this is the lower nonzero element. ∎

Consequence 5.3. *If L has a lower nonzero* (*principal*) *ideal then $I(L)$ is pure–simple.* ∎

Consequence 5.4. *A finite lattice L is pure–simple if and only if it has a smalest nonzero* (*principal*) *ideal.* ∎

REFERENCES

1. BIRKHOFF, G., – *Lattice Theory*, AMS Colloquium Publications, vol. XXV, third ed., 1967.
2. CRAWLEY, P., DILWORTH, R.P. – *Algebraic Theory of Lattices,* Prentice–Hall, Englewood Cliffs, N.J., 1973.
3. GRÄTZER, G. – *General Lattice Theory,* Akademie Verlag, Berlin, 1978.
4. HEAD, T.J. – *Purity in compact generated modular lattices,* Acta Math. Acad. Sci. Hung., 17, p.55-59, 1966.
5. STENSTROM, B. – *Rings of Quotients,* Springer Verlag, Berlin, Heidelberg, New York, 1975.

Received: 4.VI.1998

Department of Algebra
Faculty of Math. & Comput. Sci.
"Babeş–Bolyai University" of Cluj
RO-3400 Cluj–Napoca, ROMANIA
e-mail: calu@math.ubbcluj.ro