# *A new class of nil-clean elements which are exchange*

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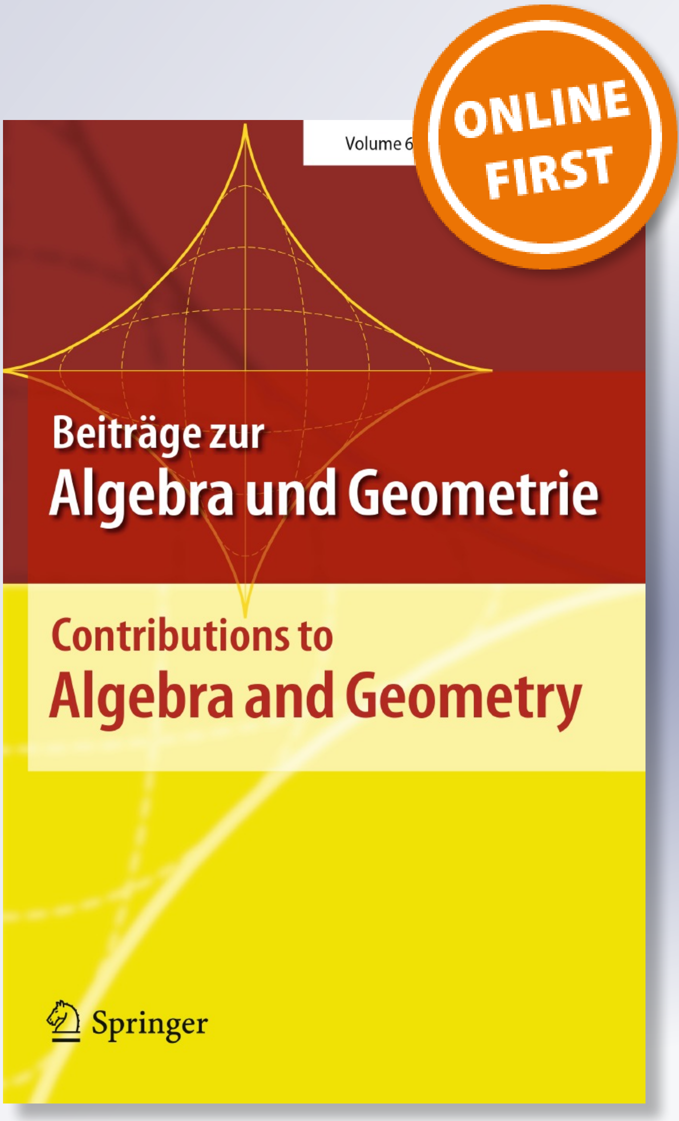

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ORIGINAL PAPER

# A new class of nil-clean elements which are exchange

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## Abstract

An element $a$ in a ring $R$ is called *left medium nil-clean* if $a = e + t$ with idempotent $e$ and nilpotent $t$, such that $et$ and $t$ commute. This proper class of nil-clean elements properly includes the strongly nil-clean elements. We show that in any unital ring, left medium nil-clean elements are exchange. Over projective-free domains we show that $2 \times 2$ and $3 \times 3$ left (or right) medium nil-clean matrices are strongly nil-clean.


**Keywords** Nil-clean element · Clean element · Exchange (suitable) element · Medium nil-clean element · Matrix rings

**Mathematics Subject Classification** 16U99 · 16U10 · 13G99 · 13H99

## 1 Introduction

An element $a$ of a ring $R$ is *nil-clean* if $a = e + t$ with idempotent $e$ and nilpotent $t$ and *strongly nil-clean* if $e$ and $t$ commute. The element $a$ is *clean* if $a = e + u$ with idempotent $e$ and unit $u$ and *strongly clean* if $e$ and $u$ commute. If all elements in a ring are nil-clean (or strongly nil-clean, or clean or strongly clean) the ring is called accordingly.

An element $a$ in a ring $R$ is called *left exchange* (or *suitable*, in the terminology of Nicholson (1977)) if there is an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. Equivalently, there is an idempotent $e \in Ra$ such that $R = Re + R(1 - a)$. Right exchange elements are defined symmetrically, but these turn out to be also left exchange. This result was observed by A.L.S. Corner in an unpublished work (see Corner 1973).

In Andrica and Călugăreanu (2014) a $2 \times 2$ integral matrix was given as an example of *nil-clean matrix which is not clean*. Since nil-clean rings are clean (Diesl 2013) and

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clean rings are exchange (Nicholson 1977), we can ask whether *there are nil-clean elements which are not exchange*.

If the nilpotent is square-zero, there are no such elements, that is

**Theorem 1** *Let $R$ be any ring, $a \in R$, and suppose that $a = e + t$ where $e^2 = e$ and $t^2 = 0$. Then $a$ is exchange in $R$.*

The proof of this theorem does not extend to nil-clean elements whose nilpotent is of index 3 (i.e. $t^3 = 0$ but $t^2 \neq 0$) or more.

However notice that, among the nil-clean elements in any ring, the strongly nil-clean elements are strongly clean and as such, are exchange.

In this paper, we define a new class of nil-clean elements which strictly includes the class of strongly nil-clean elements, for which the above proof can be adapted, not only for nilpotents of index 3, but for any nilpotents.

**Definition** An element $a$ in a ring $R$ is called *left medium nil-clean* if there exist an idempotent $e$ and a nilpotent $t$ such that $a = e + t$ and $et$ commutes with $t$, i.e. $et^2 = tet$.

Obviously

$$a \text{ strongly nil-clean } \Rightarrow a \text{ left medium nil-clean } \Rightarrow a \text{ nil-clean.}$$

As examples will show, none of these implications is reversible.

By symmetry, $a$ is *right medium nil-clean* if there exists an idempotent $e$ and a nilpotent $t$ such that $a = e + t$ and $te$ commutes with $t$, i.e. $t^2e = tet$. At least for square matrices over commutative rings it is clear that $E + T$ is left medium nil-clean iff the transpose $(E + T)^t$ is right medium nil-clean.

A great deal of research on strongly nil-clean rings (in the last 10 years or more) clarified most of the standard Ring Theory properties this class shares. However, many of these properties remain open problems when dealing with nil-clean rings. Therefore *classes in between* may be of interest.

The plan of this note is the following: in Sect. 2, we give the proof of Theorem 1 and we show that left (or right) medium nil-clean elements are exchange. In Sect. 3, under pretty general conditions, we prove that for $2 \times 2$ and $3 \times 3$ matrices, left (or right) medium nil-clean elements coincide with strongly nil-clean ones. In Sect. 4 we provide examples which show that both above inclusions are proper and, as a final remark, in contrast with the strongly nil-clean decompositions, that left (or right) medium nil-clean decompositions may not be unique. In the last section we state two open questions.

Our study is focussed on elements. A study on rings with the corresponding elements should first answer two questions: "Are nil-clean *rings* necessarily left (or right) medium nil-clean ?" and "Are left (or right) medium nil-clean *rings* strongly nil-clean ?". Since, so far, we don't know the answers, we postpone this to a future paper.

We just mention that a similar definition can be given for clean elements. However, an immediate computation shows that *left (or right) medium clean* elements are precisely the strongly clean ones.

For an idempotent $e$, $\overline{e} = 1 - e$ is the complementary idempotent, and for a square matrix $A$, $p_A(X)$ is the characteristic polynomial of $A$. A unit $u$ in a ring is called *unipotent* if $u = 1 + t$ for some nilpotent element $t$.

## 2 The main theorem

Recall that if $e$ is an idempotent and $r \in R$ is an arbitrary element, then $f = e + er\overline{e}$ is also an idempotent.

First, for Theorem 1 (see Sect. 1) we give the

***Proof*** Given $a = e + t$ with $e^2 = e$ and $t^2 = 0$, let $f = e + et\overline{e}$. Then $f^2 = f = e + et - ete = e(e + t) - et(a - t) = ea - eta \in Ra$. Therefore, in order to prove that $a$ is exchange, by condition (3), Proposition 1.1 in Nicholson (1977), it suffices to verify $Rf + R(1 - a) = R$. From $e - f = e - ea + eta = e - e(e + t) + eta = -et(1 - a) \in R(1 - a)$ we get $1 - t - f = (1 - a) + (e - f) \in R(1 - a)$. Hence $1 - t \in Rf + R(1 - a)$, so that $Rf + R(1 - a)$ contains a unit, which completes the proof. □

Next we prove our main result

**Theorem 2** *Every left medium nil-clean element is exchange.*

***Proof*** Suppose $a = e + t$ with $e^2 = e$ and $t^n = 0$. It is well-known that $1 + t$ is unipotent and $(1 + t)^{-1} = 1 - t + t^2 - \cdots + (-1)^{n-1} t^{n-1}$.

We start with the idempotent

$$f = e + e[1 - (1 + t)^{-1}]\overline{e} = e + e(t - t^2 + t^3 - \cdots + (-1)^n t^{n-1})\overline{e}.$$

In order to ease the pursuance of the computations below we denote

$$x = e(t^2 - t^3 + \cdots + (-1)^{n-1} t^{n-1}) = e[-1 + t + (1 + t)^{-1}]$$

and

$$y = et - x = e(t - t^2 + t^3 - \cdots + (-1)^n t^{n-1}) = e[1 - (1 + t)^{-1}].$$

Then $f = e + y\overline{e} = e + y - ye = e + et - x - y(a - t) = (e - y)a \in Ra$ because $yt = x$.

Further, $-y(1 - a) = -y(1 - e - t) = -y(1 - e) + x = e - f + x \in R(1 - a)$.

Therefore $1 - t + x - f = (1 - a) + (e - f + x) \in R(1 - a)$ and so $1 - t + x \in Rf + R(1 - a)$.

Notice that the computations above hold for *any* nil-clean element.

Finally $1 - t + x$ is a unit because, if $et^2 = tet$, then $-t + x$ is nilpotent. Indeed, $-t + x = [-1 + e(t - t^2 + t^3 + \cdots + (-1)^{n-1} t^{n-2})]t =: rt$ and $r, t$ commute (if $et$ and $t$ commute, then so do $et^k$ and $t$, for any $k \geq 2$). Hence $(-t + x)^n = (rt)^n = r^n t^n = 0$. □

***Remark*** With the notations in the above proof, from $f = (e-y)a$ we get $f = e(1+t)^{-1}a$ or $1-f = [1+(1+t)^{-1}]\overline{e}$ and the unipotent $v = 1-t+x = 1-t+e[-1+t+(1+t)^{-1}] = \overline{e}(1-t)+e(1+t)^{-1}$.

***Remarks*** (1) The symmetric result for right medium nil-clean elements also holds.
(2) The above proof also works if $et^2 = 0$. Indeed, then $et^k = 0$ for every $k \geq 2$ and, with above notations, $x = 0$ and $1-t$ is a unit.

## 3 2 × 2 and 3 × 3 matrices

Proofs and computation are significantly simplified if for nil-clean elements we use conjugations (i.e. similarities in the special case of matrices). As for similarity, the best result for our purpose is in Chen (2011), Proposition 11.4.9: every idempotent matrix over a *projective-free ring* admits a diagonal reduction. More precisely

**Proposition 3** *Let $R$ be a commutative ring. Then the following are equivalent:*

(1) *Every nonzero finitely generated projective $R$-module is free.*
(2) *For any idempotent $E \in \mathcal{M}_n(R)$, there exists $W \in GL_n(R)$ such that $WEW^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for some $r$.*

Here a commutative ring $R$ is *projective-free* provided that every finitely generated projective $R$-module is free. For instance, every commutative local ring and every principal ideal domain are projective-free.

Therefore, for matrix rings, in order to determine the medium nil-clean elements, by similarity, we may assume that the (nontrivial) idempotent is $E_{11}$ in $\mathcal{M}_2(R)$, and $E_{11}$ or $E_{11}+E_{22}$ in $\mathcal{M}_3(R)$, if $R$ is projective-free.

As already mentioned in the Introduction, in this section, under pretty general conditions (e. g. PID), we prove that for $2\times 2$ and $3\times 3$ matrices, left (or right) medium nil-clean elements coincide with the strongly nil-clean ones.

**Proposition 4** *Let $R$ be a projective-free domain and $A \in \mathcal{M}_2(R)$. The following are equivalent:*

(i) *$A$ is strongly nil-clean.*
(ii) *$A$ is left (or right) medium nil-clean.*
(iii) *$A$ is idempotent or nilpotent or unipotent.*

***Proof*** (iii) ⇒ (i) and (i) ⇒ (ii) being clear, for (ii) ⇒ (iii), using the similarity mentioned above, suppose $A = E_{11}+T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ with $x^2+yz = 0$, is nontrivial left (or right) medium nil-clean.

Since $T^2 = 0_2$, by medium nil-clean hypothesis, we also have $TE_{11}T = 0_2$, i.e. $\begin{bmatrix} x^2 & xy \\ xz & yz \end{bmatrix} = 0_2$. Hence, since $R$ is a domain, $x = 0$ and at least one of $y$, $z$ is zero. So nontrivial medium nil-clean matrices are of form $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix}$, that is, are idempotent matrices. The remaining trivial nil-clean matrices are of course, the nilpotents and the unipotents. □

Next, the analogue for $3 \times 3$ matrices. In the next proof we need the following special case of Proposition 3.8 from (Chen 2013) (for a reduced ring, $N(R) = 0$).

**Proposition 5** *A matrix $A \in M_3(R)$ over any reduced projective-free ring $R$ is strongly nil-clean iff*
(1) $\det A = 1$, $\det(I_3 - A) = 0$ *and* $\mathrm{Tr} A = 3$, *or*
(2) $p_A(X)$ *has a zero root,* $\det(I_3 - A) = 0$ *and* $\mathrm{Tr} A = 2$, *or*
(3) $p_A(X)$ *has a root* $= 1$, $\det A = 0$ *and* $\mathrm{Tr} A = 1$, *or*
(4) $\det A = 0$, $\det(I_3 - A) = 1$ *and* $\mathrm{Tr} A = 0$.

**Theorem 6** *Let $A = E + T$ be a left medium nil-clean $3 \times 3$ matrix over any reduced projective-free ring $R$. Then $A$ is strongly nil-clean.*

***Proof*** According to Proposition 3, we may suppose that the (nontrivial) idempotent is $E_{11}$ or $E_{11} + E_{22}$. As in Călugăreanu (2016), we denote the nilpotent $T = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$ with a $2 \times 2$ matrix $U = (u_{ij})_{i,j=1,2}$, $t = \mathrm{Tr}(U) = u_{11} + u_{22}$ and $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$, $\beta = \begin{bmatrix} x & y \end{bmatrix}$. Then we know

$\mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\alpha) = \beta\alpha$, $\mathrm{Tr}(U^2) = \mathrm{Tr}^2(U) - 2\det(U)$ and
(a) $\beta\alpha = ax + by = \det(U) - \mathrm{Tr}^2(U)$
(b) $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U)$.

Here (a) + (b) are equivalent to $\det(T) = \mathrm{Tr}(T^2) = 0$ ($\mathrm{Tr}(T) = 0$ holds as $t = \mathrm{Tr}(U)$). That is, for a given $U$, such a nilpotent $T$ exists iff there are $a, b, x, y$ satisfying (a) + (b).

**(i)** With $E_{11}$ as idempotent.

$$E_{11}\begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}^2 = E_{11}\begin{bmatrix} U^2 + \alpha\beta & (U - tI_2)\alpha \\ \beta(U - tI_2) & \beta\alpha + t^2 \end{bmatrix} = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix} E_{11} \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}.$$

By computation we get $u_{11}u_{21} = u_{12}u_{21} = u_{21}a = u_{11}x = u_{12}x = xa = 0$, $u_{12}u_{22} + ay = 0 = -(u_{11} + u_{22})a + u_{12}b$.

Then (a) is $by = \det(U) - \mathrm{Tr}^2(U) = u_{11}u_{22} - (u_{11} + u_{22})^2$ and (b) is $-byu_{11} = \mathrm{Tr}(U)\det(U) = (u_{11} + u_{22})u_{11}u_{22}$.

Notice that $u_{11} \neq 0$ is not possible: otherwise, we cancel it from (b) and (using also (a)) obtain $by = -u_{11}u_{22} - (u_{11}^2 + u_{22}^2) = -u_{11}u_{22} - u_{22}^2$, i.e. $-u_{11}^2 = 0$, a contradiction.

We go into two cases:

**Case 1**. $x = u_{21} = 0$; since $u_{11} = 0$ we have $\det(U) = 0$, $\mathrm{Tr}(U) = u_{22}$ and $by = -u_{22}^2$ (i.e. $\det \begin{bmatrix} u_{22} & b \\ y & -u_{22} \end{bmatrix} = 0$), $u_{12}u_{22} + ay = 0 = -u_{22}a + u_{12}b$. Hence matrices of the type $T = \begin{bmatrix} 0 & u_{12} & a \\ 0 & u_{22} & b \\ 0 & y & -u_{22} \end{bmatrix}$ and so $A = \begin{bmatrix} 1 & u_{12} & a \\ 0 & u_{22} & b \\ 0 & y & -u_{22} \end{bmatrix}$, for which

$\det(A) = 0$, $\mathrm{Tr}(A) = 1$ and $p_A(X)$ has 1 as a root. So these are strongly nil-clean by (3), Proposition 5.

**Case 2**. At least one of $x, u_{21}$ is not zero; then $u_{11} = u_{12} = a = 0$, $\det(U) = 0$, $t = \mathrm{Tr}(U) = u_{22}$ and so (by (a)), $by = -u_{22}^2$.

In this case, the SE $2 \times 2$ minor is zero, and the first row is zero, so we get matrices of type $A = \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & u_{22} & b \\ x & y & -u_{22} \end{bmatrix}$. Once again, $\det(A) = 0$, $\mathrm{Tr}(A) = 1$ and $p_A(X)$ has 1 as a root. So these are strongly nil-clean by (3), Proposition 5.

**(ii)** With $E_{11} + E_{22}$ as idempotent.

$(E_{11}+E_{22})\begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}^2 = (E_{11}+E_{22})\begin{bmatrix} U^2+\alpha\beta & (U-tI_2)\alpha \\ \beta(U-tI_2) & \beta\alpha+t^2 \end{bmatrix} = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$
$(E_{11}+E_{22})\begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$.

Now the computation gives: $ax = ay = at = bx = by = bt = 0$ (here $t = u_{11} + u_{22}$) and $u_{11}x + u_{22}y = u_{12}x + u_{22}y = xa + yb = 0$. We have to add (a) $\beta\alpha = ax + by = \det(U) - \mathrm{Tr}^2(U)$ and (b) $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U)$, which give $\mathrm{Tr}(U) = \det(U) = 0$, so $U$ is nilpotent and $t = 0$. Therefore $A = \begin{bmatrix} U + I_2 & \alpha \\ \beta & 0 \end{bmatrix}$ has $\mathrm{Tr}(A) = \mathrm{Tr}(I_2) = 2$ (or $= 0$ if $\mathrm{char}(R) = 2$), so now we are in case (2) or (4) Proposition 5. However, in both possible cases below, $\det(A - I_3) = 0$ so we actually are only in case (2) of the proposition. If $\alpha$ or $\beta$ is zero, then $p_A(X)$ has a zero root.

**Case 1**. $a = b = 0$; since $t = \mathrm{Tr}(U) = 0$, this gives nilpotents with 3-rd column zero and NW minor zero. Then $A$ has the 3-rd column zero and so $\mathrm{Tr}(A) = 2$, $\det(A) = 0$ and $p_A(X)$ has a zero root (here $A = \begin{bmatrix} U + I_2 & \mathbf{0} \\ \beta & 0 \end{bmatrix}$). As noticed above, $\det(I_3 - A) = 0$, holds because $I_3 - A = \begin{bmatrix} I_2 - U - I_2 & \mathbf{0} \\ -\beta & 1 \end{bmatrix}$ and so $\det(I_3 - A) = \det(I_2 - U - I_2) = \det(-U) = 0$. So these matrices are strongly nil-clean.

**Case 2**. At least one of $a, b$ is not zero; then $x = y = 0$ so $\beta = \mathbf{0}$ and $p_A(X)$ has a zero root.

As above, it remains to check $\det(I_3 - A) = 0$. Indeed: $A = \begin{bmatrix} U + I_2 & \alpha \\ \mathbf{0} & 0 \end{bmatrix}$ and $I_3 - A = \begin{bmatrix} I_2 - U - I_2 & \alpha \\ \mathbf{0} & 1 \end{bmatrix}$, which has zero determinant. So all matrices are strongly nil-clean and the proof is complete. $\square$

## 4 Examples

First we give examples which show that the inclusions mentioned in the Introduction are proper.

In what follows, a nil-clean element is called *non-trivial* if the idempotent in the decomposition is not trivial (i.e. $e \neq 0, 1$). Equivalently, such an element is neither nilpotent nor unipotent.

***Example*** For any projective-free ring $R \neq 0$ consider the *nil-clean* matrix $A = e + t = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. Since $A$ is not idempotent, not nilpotent nor unipotent *is not left (nor right) medium nil-clean* by Proposition 4.

For $R = \mathbb{Z}$, that is for $2 \times 2$ integral matrices, using Theorem 6 from (Călugăreanu 2019) (a characterization of nontrivial nil-clean matrices by Diophantine equations) and computer aid we can find all the nil-clean decompositions of $A$.

For completeness we state here this characterization

**Theorem 7** *A* $2 \times 2$ *integral matrix* $A$ *is nontrivial nil-clean iff* $A$ *has the form* $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ *for some integers* $a, b, c$ *such that* $\det(A) \neq 0$ *and the system*

$$x^2 + x + yz = 0 \tag{1}$$

$$(2a+1)x + cy + bz = a^2 + bc \tag{2}$$

*with unknowns* $x, y, z$, *has at least one solution over* $\mathbf{Z}$. *We can suppose* $b \neq 0$ *and if* (2) *holds,* (1) *is equivalent to*

$$bx^2 - (2a+1)xy - cy^2 + bx + (a^2 + bc)y = 0. \tag{3}$$

Here the nontrivial idempotent was denoted $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$.

For $a = 0$, $b = 1$, $c = -1$ Eq. (3) is: $x^2 - xy + y^2 + x - y = 0$ with (according to computer) only three solutions $(0, 0)$, $(-1, 0)$ and $(0, 1)$.

Equation (2) is: $x - y + z = -1$ which gives $z = -1 - x + y$, that is, this matrix has precisely three different nil-clean decompositions, namely

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

For each decomposition we have $tet = -t \neq 0 = et^2 = t^2e$, so this is indeed *not* left nor right medium nil-clean.

This also shows that $e + t$ is also not strongly nil-clean, but this also follows from Corollary 3.10 (Diesl 2013): $A = e + t = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ is a unit which is not unipotent.

As for the second example, of *left medium nil-clean element which is not strongly nil-clean*, the results proved in the previous section show that we cannot expect to find such examples as $2 \times 2$ or $3 \times 3$ matrices over projective-free domains.

The example of nil-clean element which is not clean given in Andrica and Călugăreanu (2014) was found in a hard way (because no computer aid for solving Diophantine equations was used).

In the last two years or so, another idea for finding such examples was brought out: to consider *special subrings* of matrix rings (e.g Wu et al. 2018). One should indeed expect to find simpler examples because in such subrings we have less clean elements.

Let $a$ be an element in a ring $R$. It is easy to check that $C_a = \begin{bmatrix} R & aR \\ aR & R \end{bmatrix}$ is a subring (with identity) of $M_2(R)$.

In the sequel we take $R = \mathbb{Z}_{n^2}$ and $a = n$, i.e. $C_n = \begin{bmatrix} \mathbb{Z}_{n^2} & n\mathbb{Z}_{n^2} \\ n\mathbb{Z}_{n^2} & \mathbb{Z}_{n^2} \end{bmatrix}$ and we consider $E = E_{11}$, $T = \begin{bmatrix} n & ny \\ nz & -n \end{bmatrix} = n \begin{bmatrix} 1 & y \\ z & -1 \end{bmatrix}$ for any $y, z \in \mathbb{Z}_{n^2}$. Then clearly $E$ is idempotent and $T$ is zerosquare, so that $A = E + T = \begin{bmatrix} 1+n & ny \\ nz & -n \end{bmatrix}$ is nil-clean (for any $y, z$).

Further, $TET = n^2 \begin{bmatrix} 1 & y \\ z & yz \end{bmatrix} = 0_2$ while *all these matrices are left (and right) medium nil-clean*.

This nil-clean *decomposition* is not strongly nil-clean (if at least one of $y, z$ is not zero, $ET = n \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} \neq n \begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix} = TE$) but this does not guarantee that $A$ is not strongly nil-clean.

To give the missing example, we consider the subring with minimum number of elements $C_2 = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ and $A = E_{11} + T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ over $\mathbb{Z}_4$ (which from above we know as left - or right - medium nil-clean).

Since strongly nil-clean elements are strongly clean (in any ring) it suffices to show that $A$ is not strongly clean.

First notice that for any matrix $M = \begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix} \in C_2$, $\det(M) = ad \in U(\mathbb{Z}_4) = \{1, 3\}$ iff $a, d \in \{1, 3\}$.

Next, since $M^2 = \begin{bmatrix} a^2 & 2b(a+d) \\ 2c(a+d) & d^2 \end{bmatrix}$, $M$ is idempotent only if $a, d$ are idempotent, that is, $a, d \in \{0, 1\}$ in $\mathbb{Z}_4$.

Finally we find all the clean decompositions for $A$ and check that none is strongly clean. For any unit $V$ we should have idempotent $A - V$. Since for idempotent $M = \begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix}$ we must have $a, d \in \{0, 1\}$ (and $a, d \in \{1, 3\}$ for units $V$) it is readily seen that $V = \begin{bmatrix} 3 & * \\ * & 1 \end{bmatrix}$, that is four possibilities.

All 4 give clean decompositions:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}.$$

None has commuting idempotent-unit.

Finally, recall a well-known property of *strongly nil-clean elements*: these *have precisely one strongly nil-clean decomposition*.

This does *not* happen for medium nil-clean elements as our above example shows.

Indeed, $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is another left and right medium nil-clean decomposition.

## 5 Open questions

Since strongly nil-clean elements are strongly clean, left (or right) medium nil-clean elements are exchange and nil-clean rings are clean, a natural question is whether *left (or right) medium nil-clean elements are clean*.

Notice that the second example in the previous section *suggests an afirmative answer*: the matrix $A$ is left (and right) medium nil-clean, not strongly nil-clean but clean.

As usual there is another natural question: *are left medium nil-clean elements also right medium nil-clean*?

Clearly, we can answer this question at ring or at element levels.

At element level, a negative answer needs an *example of left medium nil-clean element which is not right medium nil-clean*.

The results proved in Sect. 3 show that we cannot expect such examples in $2 \times 2$ or $3 \times 3$ matrix rings over projective-free rings (where left or right medium nil-clean are also strongly nil-clean).

**Added in proof**. As T.Y. Lam observed, the left medium nil-clean property can be further generalized, still preserving the exchange property (with a similar proof): requiring $et^2 \in Ret$ instead of $et^2 = tet$.

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