MATHEMATICA, Tome 37(60), N^{os} 1–2, 1995, pp. 55–57

NOTE ON A MODULE-THEORETIC EXERCISE

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In his well-known book on rings and modules, R. Wisbauer states the following exercise ([6], p.264): *Show that the following are equivalent for a Z-module M: (a) M is locally artinian: (b) M has dcc for cyclic submodules: (c) M has essential socle: (d) M is a torsion module.* In thid short note we generalize this exercise in an algebraic modular lattice.

1. Introduction

In what follows for a complete lattice L we will use the following definitions: L is **artinian** [**noetherian**] if it satisfies the **dcc** [**acc**] (the descending [ascending] chain condition), **locally artinian** if all its compact elements are artinian (an element $a \in L$ has the property P if the sublattice $a/0$ has P), is H-noetherian (see [4]) if the compact elements form an ideal (or equivalently, if for each $a \leq c$, with c compact, a is also compact), is **cyclic** if it is distributive and noetherian, has (**RSC**) (the restricted socle condition) (see [2]) if for each $a \in L$, a essential in L, $1/a$ has atoms and is **torsion** (see [1]) if for each $a \in L$, $a \neq 1$ the sublattice $1/a$ has atoms. For all the notions and notation we refer to [3] and [5].

According to the above-mentioned exercise we are interested in the following conditions in a lattice L: (a) is locally artinian; b) L has *dcc* for cyclic elements; (c) the socle $s(L)$ is essential in L; (d) is torsion. For simplification, let L be an upper continuous modular lattice.

2. Results

PROPOSITION 2.1. *If L has* (**RSC**) *then* (*c*) *and* (*d*) *are equivalent.*

Proof. The socle of a torsion lattice is essential. Indeed, let $0 \neq a \in L$ and b a pseudocomplement of a in L. Surely $b \neq 1$ so that, L being torsion, $1/b$ contains an atom c. Using the modularity, one shows that $a \wedge c$ is an atom in L. But $0 \neq a \wedge c \leq a \wedge s(L)$ so that the socle $s(L)$ is essential in L. Conversely, from [1] we know that in an (**RSC**) lattice $s(L)$ is essential in L iff $1/s(L)$ is torsion. The sublattice $s(L)/0$ being torsion, L is also torsion. □

In order to relate (a) and (c) we need the following

LEMMA 2.1. *In an upper continous lattice L the noetherian elements coincide to the compact elements iff L is H-noetherian.*

Proof. If $a \leq c$, with c compact, by hypothesis c is noetherian and so a is noetherian too. Hence (again by hypothesis) a is compact and L is H-noetherian. Conversely, each noetherian element is compact (see[3]), so that let c be a compact element in L. If L is H-noetherian each $a \leq c$ is compact too. But then c is noetherian (see [3]) so that compact and noetherian elements coincide. □

PROPOSITION 2.2. *A H-noetherian algebraic locally artinian lattice L is atomic.*

Proof. In a H-noetherian locally artinian lattice the compact elements are noetherian and artinian and hence of finite lenght (have composition series). In this case each compact element contains an atom so that L being algebraic each non-zero element contains an atom. □

Consequence 2.1. *In a H-noetherian algebraic lattice L(a) implies (c).*
Indeed, obviously the socle of an atomic lattice L is essential in L. □

LEMMA 2.2. *Every noethernian torsion lattice is artinian.*

Proof. Let $a \in L$ be maximal with the property that $a/0$ is artinian. If $a \neq 1$, the lattice being torsion $1/a$ has an atom b. We show that $b/0$ is artinian and hence contradict the maximality of a. If $c_1 \geq c_2 \geq .. \geq c_n \geq ..$ is a descending chain in $b/0$ and for a $m \in \mathbf{N}$, $c_m \leq a$ then the chain must be finite. If for every $n \in \mathbf{N}$, we don't have $c_n \leq a$ then $a \vee c_n = b$ and, by modularity, if the chain $(c_n)_{n \in \mathbf{N}}$ is not finite, neither is finite the chain $(a \wedge c_n)_{n \in \mathbf{N}}$. But this contradicts $a/0$ artinian. □

PROPOSITION 2.3. *In a H-noetherian lattice* (d) *implies* (*a*).

Proof. If a lattice L is H-noetherian, each compact element c is already noetherian so if L is also torsion, the sublattice $c/0$ is also torsion (true for modular, pseudocomplemented lattices, see [1]). By the above lemma, every noetherian torsion lattice is artinian, so c is artinian and hence L is locally artinian. □

Consequence 2.2. *In a algebraic H-noetherian* **(RSC)** *modular lattice the conditions (a), (c) and (d) are equivalent.* □

Finally, the condition (b) is more difficult to relate to the other three conditions. We need

LEMMA 2.3. *If an algebraic lattice L has* **dcc** *for cyclic elements then L has also the* **dcc** *for compact elements.*

Proof. In an upper continuous lattice L the set of all the elements a such that $a/0$ satisfies the **dcc** for compact elements (in an algebraic lattice these are the compact elements from L that belong to $a/0$), Zorn's Lemma is applicable so that let m be maximal in this set. If $m \neq 1$ in $1/m$ let c be a minimal cyclic element (1, if m is maximal in L). It is shown that $c/0$ satisfies also the **dcc** for compact elements, contradicting the maximality of m (see also [6]). □

LEMMA 2.4. *If an algebraic lattice L has the* **dcc** *for compact elements for each compact element c ∈ L the sublattice c/0 is supplemented.*

Proof. Let c be a compact element and $a \leq b \leq c$. By the **dcc** let s be a minimal compact such that $a \vee s = b$. Then s is a supplement of a in $b/0$ (see also [6]). □

PROPOSITION 2.4. *In an algebraic H-noetherian lattice the condition* (*b*) *implies the condition* (*a*).

Proof. As above we obtain the condition (a) if every sublattice $c/0$ is torsion for each compact c. Using the last two lemmas the only supply left is: every supplemented lattice is torsion. But this is readily checked in our hypothesis. □

3. Final comments

The implication (a) $\Rightarrow$(b) seems to lead to complicated restrictions related to cyclic elements that will make the object of another study.

As we have seen in the above three propositions the equivalence of the conditions (a), (b), (c) and (d) is far more general as only for abelian groups (**Z**-modules). Once more time the utility of the condition (**RSC**) and of the *H*-noetherianity in this context is evident.

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Received 25 IV 1995

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