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Nil-clean integral 2×2 matrices: The elliptic case

by
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Abstract

Nil-clean integral 2×2 matrices are investigated starting with a characterization which involves a Diophantine second degree equation. Mainly similarity classes of matrices A with $1 - 4\det(A) < 0$ are determined (that is, the elliptic case). The strongly nil-clean matrices are completely determined and a large class of uniquely nil-clean matrices is found. In particular, a class of uniquely nil-clean matrices which are not strongly nil-clean is found.

Key Words: Uniquely nil-clean, strongly nil-clean, clean, quadratic Diophantine equation.
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1 Introduction

The important rôle of idempotents, nilpotent elements and units in Ring Theory was recognized already a century ago. Considering elements which are sums of two such elements is more recent. Sums of an idempotent and a unit (called *clean* elements) were defined by Nicholson (1977) in [9]. Sums of an idempotent and a nilpotent element (called *nil-clean* elements) were considered by Diesl (2006) in his Ph. D. thesis, and finally sums of a unit and a nilpotent element (called *fine* elements) were considered by the author and T. Y. Lam (2015) in [5]. Further, a ring (with identity) was called *clean* if all its elements are clean, *nil-clean* if all its elements are nil-clean and *fine* if all its nonzero elements are fine. An element was called *uniquely* clean (or nil-clean or fine) if it has only one clean (or nil-clean, or fine) decomposition, and strongly clean (or nil-clean or fine), if the components of the decomposition commute. If $a = e + t$ with $e^2 = e$ and nilpotent t, we say that e is *the* idempotent in this nil-clean decomposition.

Analogously, uniquely or strongly clean (or nil-clean or fine) rings were defined and between all these classes, several inclusions were (easily) established.

Nil-clean rings are clean, uniquely clean rings are Abelian (i.e., the idempotents are central; see [10]) and so strongly clean, uniquely nil-clean rings are Abelian (see [7]) and so strongly nil-clean, fine rings are simple.

Despite all these inclusions, when it comes to compare the corresponding types of elements, everything fails. The ring of all the 2×2 integral matrices seems to contain all the counterexamples one could think of!

In [2], the matrix $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ was shown to be nil-clean but not clean (surprisingly, it turns out that our example is also an example of uniquely nil-clean element in $\mathcal{M}_2(\mathbf{Z})$

which is not strongly nil-clean.). Recently, in [8], the matrix $\begin{bmatrix} 8 & 3 \\ 0 & 0 \end{bmatrix}$ was shown to be uniquely clean but not strongly clean. Since nonzero elements are strongly fine iff these are units, it is easy to give examples of uniquely fine elements which are not strongly fine: nontrivial idempotents in $\mathcal{M}_2(\mathbf{F}_2)$, the ring of the 2×2 matrices over the 2-element field.

Since in the bibliography known to the author, examples of uniquely nil-clean elements are scarce, the initial goal of this paper was to give such examples and especially, to give an example of uniquely nil-clean element which is not strongly nil-clean. Along the way, we succeeded far more: to determine, up to similarity, all nil-clean *integral* 2×2 matrices A such that $1 - 4\det(A) < 0$, and in particular, the strongly nil-clean matrices and also a large class of uniquely nil-clean matrices.

As in [2], the problem can be reduced to a Diophantine second degree equation which, when searching for nil-clean matrices, is in the elliptic or the hyperbolic case (parabolic case is not possible). The condition above, $1 - 4\det(A) < 0$, characterizes the elliptic case.

As the title already shows, in this paper we investigate only the elliptic case (the hyperbolic case, which is far more difficult will be the subject of a future study).

In Section 2 we recall some important results obtained by Behn and Van der Merwe in [4], useful for our approach, and a characterization for nontrivial nil-clean matrices is given.

In Section 3 we first describe all the trivial nil-clean matrices (i.e. matrices whose decomposition uses a trivial idempotent - that is, 0 or 1 - or the zero nilpotent). Then, using as idempotent for nil-clean matrix decompositions the matrix unit E_{11}, we determine, up to similarity, all strongly nil-clean matrices. In Section 4, classes of not nil-clean matrices with not fundamental discriminant are emphasized, and we prove that matrices of type $\begin{bmatrix} 0 & -\delta \\ 1 & 1 \end{bmatrix}$ have nil-clean index (precisely) 3. In section 5 we show that matrices of type $\begin{bmatrix} 3k+2 & -3(3k+2) \\ 9k+5 & -3k-1 \end{bmatrix}$ are uniquely nil-clean (and not strongly nil-clean).

For a square matrix A, A^T denotes the transpose; E_{ij} denotes the matrix (i,j)-unit, that is the $n \times n$ matrix whose entries are all zero excepting the (i,j)-entry, which is 1.

2 Nil-clean 2×2 matrices and similarity

Definition. Two 2×2 matrices A, B over any unital ring R, are *similar* (or *conjugate*) if there is an invertible matrix U such that $B = U^{-1}AU$. Since similarity is obviously an equivalence relation, a partition of $\mathcal{M}_2(R)$ corresponds to it. The subsets in this partition are called *similarity classes*.

Such classes may consist only in one matrix: a matrix A forms a singleton class iff $AU = UA$ for every invertible matrix U (e.g. 0_2, I_2 or every scalar matrix.

If A is nilpotent (or idempotent) and B is similar to A then B is also nilpotent (resp. idempotent). This similarity invariance clearly extends to nil-clean matrices and it also restricts to uniquely or strongly nil-clean matrices, respectively. Rephrasing, the notions of nil-clean, uniquely nil-clean and strongly nil-clean are similarity invariants. So is the nil-clean index, where an element a in a ring R has *nil-clean index* n (possibly infinite) if there exist exactly n idempotents e such that $a - e$ is nilpotent (see [3]).

Remark. If both A, B are nil-clean, these may not be similar (e.g., different determinant); more, having also the same determinant (or even the same characteristic polynomial), is still not sufficient for A and B to be similar. That is, even nil-clean matrices with the same trace and determinant may belong to different similarity classes.

In the sequel $R = \mathbf{Z}$, that is, we deal only with 2×2 integral matrices. Our main goal is to determine, in reasonable conditions, all the nil-clean, all the strongly nil-clean matrices and large classes of uniquely nil-clean matrices.

By the above paragraph, to each nil-clean matrix A we can associate infinitely many nil-clean matrices, namely all its conjugates: $U^{-1}AU$, with any unit $U \in \mathcal{M}_2(\mathbf{Z})$, that is, $U \in GL_2(\mathbf{Z})$.

Thus, to determine all nil-clean matrices actually means to find all the similarity classes of nil-clean matrices. In doing so, it is natural to choose in each similarity class a special representative. In this paper, this will be done into two different ways.

The first has already been done by Behn and Van der Merwe in [4]. In [4], an algorithm is presented, which, given a 2×2 matrix, finds a canonical representative (called reduced or primitive) in its similarity class.

The second consist in choosing in the similarity classes, a representative which uses a special idempotent (namely the matrix unit E_{11}).

As for the first, recall (from [4]) the following

Definition. A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $D = \mathrm{Tr}(A)^2 - 4\det(A) < 0$ is *reduced* if $|d-a| \leq c \leq -b$ and, $d \geq a$ if at least one is equality, i.e. $|d-a| = c$ or $c = -b$. Notice that if $|d-a| < c < -b$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ are different reduced matrices.

If D is a square (e.g. $\det(A) = 0$), that is, the characteristic polynomial of the matrix factors over the integers, say, $f(x) = (x-a)(x-d)$, where $a \geq d$, then, for $a \neq d$ the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is *reduced* if $0 \leq b < a-d$, and, for $a = d$, if $b \geq 0$.

Examples. 1) The matrix unit $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is reduced, I_2 is also (idempotent and) reduced, but E_{22} is *not reduced*. Actually E_{11} and E_{22} are similar: for $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $U^2 = I_2$ and $UE_{11}U = E_{22}$.

2) The matrix unit $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is reduced, E_{21} is (also nilpotent but) *not reduced*. Actually $E_{21} = UE_{12}U$, so these are similar.

3) The unipotent matrix $V = I_2 + E_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is reduced but $W = I_2 + E_{21} = V^T$ is (also unipotent but) *not reduced*. Again $UVU = V^T$.

Next, also from [4] (Theorems 3.3 and 5.2), recall the following results

Theorem 1. *Consider matrices A in $\mathcal{M}_2(\mathbf{Z})$ with a fixed $\mathrm{Tr}(A)$ and $\det(A)$ and with $D = \mathrm{Tr}(A)^2 - 4\det(A) < 0$. Then there is precisely one reduced matrix in each matrix similarity class.*

Theorem 2. *Let $M \in \mathcal{M}_2(\mathbf{Z})$ and assume that the characteristic polynomial of M factors over $\mathbf{Z}$. Then M is similar to a reduced matrix. Moreover, this class representative is* *unique thus no two different reduced matrices are similar.*

More precisely, if $f(x)=(x-a)(x-d)$ (and $a\geq d$) then M is similar to a matrix $\left[\begin{array}{cc} a & b \\ 0 & d \end{array}\right]$ with $0\leq b\leq |a-d|$, and for $a=d$, with $b\geq 0$.

The second way of representing the similarity classes relies on the following important consequence of the latter theorem.

Corollary 1. *Any non-trivial 2×2 idempotent integral matrix is similar to E_{11}.*

That is, all nontrivial idempotent matrices belong to the same similarity class and E_{11} is the only reduced representative in this class.

As the reviewer noticed, this result can also be obtained directly, by looking at matrices as endomorphisms of a free rank 2 Abelian group.

Examples. 1) If $E=\left[\begin{array}{cc} 1 & 0 \\ s & 0 \end{array}\right]$ then with $P=\left[\begin{array}{cc} 1 & 0 \\ s & 1 \end{array}\right]$ we get $E_{11}=P^{-1}EP$.

2) If $E=\left[\begin{array}{cc} s+1 & -s-1 \\ s & -s \end{array}\right]$ then with $P=\left[\begin{array}{cc} s+1 & 1 \\ s & 1 \end{array}\right]$ we get $E_{11}=P^{-1}EP$.

Analogously

Corollary 2. *Any 2×2 nilpotent integral matrix is similar to kE_{12} for some integer $k\geq 0$.*

and

Corollary 3. *Any 2×2 unipotent integral matrix is similar to $V_k=\left[\begin{array}{cc} 1 & k \\ 0 & 1 \end{array}\right]$ for some $k\geq 0$.*

As already stated in the Introduction, we restrict our investigation to matrices A such that $D=\mathrm{Tr}(A)^2-4\det(A)<0$. According to Theorem 1, for such matrices, to determine all nil-clean matrices amounts to find all reduced nil-clean matrices. Moreover, to find all uniquely (or strongly) nil-clean matrices, means to determine all the reduced uniquely (or strongly) nil-clean matrices.

Next, recall that an integer d is a *fundamental discriminant* if $d\neq 1$, d is not divisible by any square of odd number and $d\equiv 1(\mathrm{mod}4)$ or $d\equiv 8,12(\mathrm{mod}16)$. Actually d is a fundamental discriminant iff d is the discriminant of a quadratic number field. It turns out that up to isomorphism, there is exactly one quadratic field for every fundamental discriminant.

The following characterization was partly hidden in [2].

Theorem 3. *A 2×2 integral matrix A is nontrivial nil-clean iff A has the form $\left[\begin{array}{cc} a+1 & b \\ c & -a \end{array}\right]$ for some integers a,b,c such that $\det(A)\neq 0$ and the system*

$$\left\{\begin{array}{lr} x^2+x+yz=0 & (1) \\ (2a+1)x+cy+bz=a^2+bc & (2) \end{array}\right.$$

with unknowns x,y,z, has at least one solution over $\mathbf{Z}$. We can suppose $b\neq 0$ and if (2) holds, (1) is equivalent to

$$bx^2-(2a+1)xy-cy^2+bx+(a^2+bc)y=0 \quad (3).$$

Proof. Recall that any nontrivial idempotent is characterized by zero determinant and trace = 1 and any nilpotent by zero determinant and zero trace. Therefore, the nil-clean matrices we have to investigate have trace = 1 and so are of form $A = \begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$. Here $\det(A) \neq 0$ (otherwise, by Cayley-Hamilton theorem, A is idempotent and so trivial nil-clean). Such matrices should have a nil-clean decomposition $A = E + N$ with nontrivial idempotent $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ i.e., $\mathrm{Tr}(E) = 1$ and $-\det(E) = x^2 + x + yz = 0$, that is (1), and nilpotent N. Since the condition $\mathrm{Tr}(N) = 0$ is already fulfilled, using (1), the condition $\det(N) = 0$ amounts to $(2a+1)x + cy + bz = a^2 + bc$, that is (2).

We discard right away the case when both $b = c = 0$. Indeed, if so, (2) becomes $(2a+1)x = a^2$ and for integers, it is readily seen that $2a+1$ divides a^2 only for $a = 0$ and $a = -1$. However in these cases A is an idempotent and so trivial nil-clean.

Thus $b \neq 0$ (the case $c \neq 0$ is symmetric). Multiplying (1) by b and eliminating z, we get the Diophantine equation $bx^2 - (2a+1)xy - cy^2 + bx + (a^2 + bc)y = 0$, that is (3). □

Remarks. 1) The Theorem remains true for matrices over any integral domain.

2) For further use, observe that equations (1) or (3) have always the solutions $(x, y) = (0, 0)$ and $(x, y) = (-1, 0)$ with an arbitrary z. Moreover, (3) has also the solution $(x, y) = (a, b)$.

For easy reference we state here the following useful result:

Lemma 1. *The equation (2) has the solution*

(i) $(0, 0)$ *iff* b *divides* a^2*;*

(ii) $(-1, 0)$ *iff* b *divides* $(a+1)^2$*;*

(iii) (a, b) *iff* b *divides* $a^2 + a$*.*

3 Nil-clean matrices with idempotent E_{11}

A nil-clean element (in any ring) will be called *trivial* if its decomposition uses a trivial idempotent or the zero nilpotent. That is, the trivial nil-clean elements are the nilpotent elements, the unipotent elements and the idempotents. All the other nil-clean elements will be called *nontrivial*.

Using the procedure described in the previous section it is easy *to determine* all the trivial nil-clean matrices (that is, their similarity classes).

The nilpotent matrices: the singleton class $\{0_2\}$ or the classes represented by kE_{12}, $k \geq 0$.

The unipotents (i.e. matrices $I_2 + N$ with nonzero nilpotent N): the classes are represented by $V_k = I_2 + kE_{12}$, $k \geq 0$.

The idempotents: the singleton classes $\{0_2\}$, $\{I_2\}$ or the class represented by E_{11}.

Proposition 1. *In any ring, all trivial nil-clean elements are strongly nil-clean. In* $\mathcal{M}_2(\mathbf{Z})$*, all trivial nil-clean matrices are uniquely nil-clean, excepting the nontrivial idempotents.*

Proof. Since 0_2 and I_2 are obviously uniquely nil-clean, only the last statements needs justification. To prove that nonzero nilpotents and unipotents $\neq 1$, are uniquely nil-clean and that nontrivial idempotents are not uniquely nil-clean, we just have to check this for the reduced representatives in the corresponding similarity classes.

It is easily checked that kE_{12} and V_k are uniquely nil-clean, for any $k \geq 0$. Finally, $E_{11} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}$, for any $a \in \mathbf{Z}$, are (infinitely many) different nil-clean decompositions. ∎

Therefore, in the sequel, we consider *only nontrivial* nil-clean matrices.

As mentioned in the Introduction, by Corollary 1, every nontrivial nil-clean matrix is similar to one whose idempotent, in the nil-clean decomposition, is E_{11}. More precisely

Lemma 2. *A matrix* $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ *(is nil-clean and) admits E_{11} as idempotent in a (nontrivial) nil-clean decomposition iff $a^2 + bc = 0$. In this case, neither b nor c are zero.*

Proof. $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ decomposes as $E_{11} + N$ iff $N = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is nilpotent. Hence $\det(N) = -a^2 - bc = 0$, $\det(A) = -a$ and neither b nor c is zero (otherwise $a = 0$ and A is idempotent, that is trivial nil-clean). ∎

Obviously, matrices of this type are easier to handle. In equation (3) the y coefficient vanishes and (2) is equivalent to b divides $(2a + 1)x + cy$.

The next result nicely completes Proposition 1.

Theorem 4. *The only strongly nil-clean matrices in $\mathcal{M}_2(\mathbf{Z})$ are the trivial ones.*

Proof. Suppose A is nontrivial strongly nil-clean in $\mathcal{M}_2(\mathbf{Z})$. Then A is similar to a strongly nil-clean matrix B whose idempotent is E_{11}, that is, $B = E_{11} + N$ with a nilpotent $N = \begin{bmatrix} s & t \\ c & -s \end{bmatrix}$ such that E_{11} and N commute. However $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s & t \\ u & -s \end{bmatrix} = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & t \\ u & -s \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ u & 0 \end{bmatrix}$ hold iff $t = u = 0$ and so $s = 0$ (because $s^2 + tu = 0$). Hence $N = 0_2$, a contradiction. ∎

Actually, only two nil-clean matrices satisfy $a^2 + bc = 0$: $\begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mp 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix} = E_{11} + \begin{bmatrix} -1 & \pm 1 \\ \mp 1 & 1 \end{bmatrix}$ (so these are *not uniquely nil-clean*). These two matrices are indeed similar:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Observe that $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ is reduced but the transpose is not. This matrix is not strongly nil-clean as already seen in the general case above (clearly it is also not unipotent).

In general, for any given $a \leq -1$, and prime decomposition $|a| = p_1^{k_1} \cdots p_s^{k_s}$ we get $2d(a^2)$ nil-clean matrices, if $d(n)$ denotes the number of positive divisors of n, including 1 and n itself. Recall that if $n = p_1^{k_1} \cdots p_s^{k_s}$ is the prime factorization of n, then $d(n) = (k_1+1)\cdots(k_s+1)$. If i is any (integer) divisor of a^2 and $a^2 = ij$, the corresponding nil-clean matrices are $\begin{bmatrix} a+1 & \pm i \\ \mp j & -a \end{bmatrix}$ and transposes (if $i = j = a$, these coincide with transposes).

Analogously with results in Section 2, we have the following

Proposition 2. *If $a^2 + bc = 0$ and $|a|$ is a prime then $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ has index at least 3 and is not strongly nil-clean.*

Proof. Since $|a|$ is a prime, the equation $a^2 + bc = 0$ has six different integer solutions. Therefore we have six nil-clean matrices, namely: $\begin{bmatrix} a+1 & \pm 1 \\ \mp a^2 & -a \end{bmatrix}$, two transposes and $\begin{bmatrix} a+1 & \pm a \\ \mp a & -a \end{bmatrix}$. All clearly decompose with E_{11} and all must be similar. We can choose a not reduced representative $\begin{bmatrix} a+1 & a \\ -a & -a \end{bmatrix}$, which is not uniquely nil-clean:

$\begin{bmatrix} a+1 & a \\ -a & -a \end{bmatrix} = E_{11} + \begin{bmatrix} a & a \\ -a & -a \end{bmatrix} = \begin{bmatrix} a+1 & a+1 \\ -a & -a \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+1 & a \\ -a-1 & -a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, nor strongly nil-clean (by Theorem 4). ∎

4 More results

By taking $2a + 1 = b = -c$, because of (2), we can generate many (reduced) *not nil-clean* matrices (with trace = 1).

Proposition 3. *For any $k \geq 1$, the matrices $M_k = \begin{bmatrix} 1-k & 1-2k \\ 2k-1 & k \end{bmatrix}$ are not nil-clean and have not fundamental discriminant.*

Proof. By definition these matrices are reduced and equation (2) is

$$-(2k-1)x + (2k-1)y - (2k-1)z = (k-1)^2 - (2k-1)^2 = -3k^2 + 2k.$$

Now $\gcd(3k-2; 2k-1) = 1$ since $3(2k-1) - 2(3k-2) = 1$, and so are $\gcd(k; 2k-1) = 1$. Therefore (2) has no (integer) solutions.

Here $D = 1 - 4(3k^2 - 3k + 1) = -3(2k-1)^2$ so indeed it is not a fundamental discriminant. ∎

Remark. Less suffices for the discriminant to be not fundamental.

If $\gcd(2a+1; b; c) = d \neq 1$ then $D = 1 - 4\det(A) = (2a+1)^2 + 4bc$ is divisible by the odd square d^2.

This class of examples could suggest that *a matrix with trace =1 and fundamental discriminant is nil-clean.* This fails, as the following example shows.

Example. $G = \begin{bmatrix} -3 & -7 \\ 17 & 4 \end{bmatrix}$, so $a = -4$, $b = -7$, $c = 17$; $\det(G) = 119 - 12 = 107$ and $D = 1 - 4\det(G) = -427 = 7 \times 61$ is fundamental, class 2. Equation (3) $-7x^2 + 7xy - 17y^2 - 7x - 103y = 0$ has exactly the 3 solutions in Lemma 1. Now $7 \nmid 16$, $7 \nmid 12$ and $7 \nmid 9$, all $(0,0)$, $(-1,0)$ and (a,b) are eliminated, so G is not nil-clean (alternatively, (2) is not verified: $-7x + 17y - 7z = -103$).

Computer verifications may suggest a nil-clean index for matrices of a given form (i.e. given determinant and entry a), but it is not easy to prove that this number is indeed the nil-clean index. Here is a sample.

Theorem 5. *Matrices of type* $\begin{bmatrix} 0 & -\delta \\ 1 & 1 \end{bmatrix}$ *with* $\delta \geq 1$ *have exactly three nil-clean decompositions.*

Proof. It is easy to see that the index is at least 3: $\begin{bmatrix} 0 & -\delta \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\delta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\delta \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1-\delta \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. In the remaining of the proof we show that it is precisely 3.

Since the Diophantine equation (3) is simpler if the idempotent is E_{11} (the y coefficient is zero) we first pass (conjugation with $\begin{bmatrix} 1 & \delta-1 \\ 0 & 1 \end{bmatrix}$; see example 1 in Section 2) to the corresponding (similarity) representative, which is $B = \begin{bmatrix} 1-\delta & -\delta^2 \\ 1 & \delta \end{bmatrix}$. The Diophantine equation (3) (multiplied by -1) is $\delta^2 x^2 - (2\delta - 1)xy + y^2 + \delta^2 x = 0$, which can be written

$$y(x + y) = \delta x[2y - \delta(x + 1)] \qquad (3').$$

Equation (2) is $(1 - 2\delta)x + y - \delta^2 z = 0$, which can be written

$$x + y = \delta(2x + \delta z) \qquad (2').$$

Since (3) represents an ellipse, using partial derivatives, it is readily checked that $-\delta \leq x \leq 0$, $-\delta^2 \leq y < \frac{\delta^2}{4\delta-1}$ and that $(-\delta, -\delta^2)$ is the minimum point on the ellipse (for concrete graphs use e.g. [6]).

To simplify the proof, the case $\delta = 1$ will be excepted. In this case, for $V = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ we already have $a^2 + bc = 0$, and $V = E_{11} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Moreover $\det V = 1$, so this is a unit (which is not unipotent). Equation (3) is $x^2 - xy + y^2 + x = 0$, which has precisely three solutions: $(0,0)$, $(-1,0)$ and $(a,b) = (-1,-1)$. All satisfy (2) (e.g. see Lemma 1), so we have exactly three nil-clean decompositions, the one above and $V = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

From (2'), $x+y=-k\delta$, a parallel line to the 2-nd bisector, so in order to intersect the ellipse we need $0 \leq k \leq \delta+1$. It is readily checked that $k \in \{0,2,\delta\}$ give the *three* nil-clean decompositions of B, we already know (that is, corresponding to the solutions $(0,0)$, $(-1,1-2\delta)$ and $(-\delta,\delta-\delta^2)$). One can also check that $(-1,0)$ and $(a,b)=(-\delta,-\delta^2)$ do not satisfy (2): $-1=\delta(-2+\delta z)$ for the first ($\delta \neq 1$) and $2\delta-1=\delta(z-1)$ impossible since $\gcd(\delta;2\delta-1)=1$.

In the sequel we show that the system (2') + (3') has no (integer) solutions for $k \in \{1,3,4,...\delta-1,\delta+1\}$ and so B has exactly three nil-clean decompositions. We can easily discard also the case $k=\delta+1$. In this case the intersection of the line $x+y=-\delta(\delta+1)$ is (only) the minimum point of the ellipse, i.e. $(-\delta,-\delta^2)$, which we already saw not satisfying (2).

Next, $x+y=-k\delta=\delta(2x+\delta z)$ implies $-k=2x+\delta z$ and so δ divides $-k-2x$. Now $-\delta \leq x \leq 0$ implies $-k \leq -k-2x \leq 2\delta-k$ and since $1 \leq k \leq \delta$, we obtain $-k-2x=0$ or $-k-2x=\delta$, with one exception: if $k=\delta$ then $-\delta \leq -k-2x \leq 2\delta-\delta=\delta$, i.e. $-\delta=-k-2x$ is also possible.

In the first case, $z=0$ and so $x(x+1)=0$ from (1). Here $x=0$ implies $k=0$ and $x=-1$ implies $k=2$, not in our range.

In the second, we replace $x+y=-k\delta$ (and $y=-x-k\delta$) in (3'), obtaining $(\delta+2)x^2+[k+\delta(2k+1)]x+k^2\delta=0$. Further, if we multiply by 4 and replace $2x=-k-\delta$, we get $\delta(\delta-k)^2=0$, which has solutions only for $k=\delta$ ($\delta \neq 0$) and this is $x=-\delta$, with $(\delta,\delta-\delta^2)$ the (third) solution for (2)+(3).

Finally, if $k=\delta$ and $-\delta=-k-2x$, then $x=0$ is the only solution, but $(0,-\delta^2)$ does not verify (3') and the proof is complete. □

Inspection of the example $\det(A)=57$, $a=2$ suggests that for a given $\det(A)$ and entry a, *the nil-clean index increases with the difference* $||b|-|c||$. Indeed, for $(b,c)=(9,-7)$ we had index 1, for $(b,c)=(3,-21)$ index 2, and for $(b,c)=(1,-63)$ index 3. Moreover, for $\det(A)=57$, $a=3$, $(b,c)=(23,-3)$ has index 2 and $(b,c)=(69,-1)$ has index 3. Unfortunately *this is false* as shown by the following

Example. All four matrices $\begin{bmatrix} -5 & -36 \\ 1 & 6 \end{bmatrix}$, $\begin{bmatrix} -5 & -18 \\ 2 & 6 \end{bmatrix}$, $\begin{bmatrix} -5 & -12 \\ 3 & 6 \end{bmatrix}$ and $\begin{bmatrix} -5 & -6 \\ 6 & 6 \end{bmatrix}$ have index 3**.** We just mention that for $\det(A)=6$, the discriminant is $D=-23$, which is class 3.

Would this be true, we should have, combining with the previous Proposition) a proof for the following

Conjecture 1. *Nontrivial nil-clean matrices (in $\mathcal{M}_2(\mathbf{Z})$) have nil-clean index at most 3.*

In trying to find a nil-clean matrix of index (at least) 4, one should try with (nil-clean) matrices of higher class.

Here is a sample: $D=-479$ is class 25, and so $\det(A)=120$. By the previous Proposition we already know that $\begin{bmatrix} 0 & -120 \\ 1 & 1 \end{bmatrix}$ has index exactly 3. Further $\begin{bmatrix} 0 & -60 \\ 2 & 1 \end{bmatrix}$ has index 2, $\begin{bmatrix} 0 & -30 \\ 4 & 1 \end{bmatrix}$ has index 2, $\begin{bmatrix} 0 & -15 \\ 8 & 1 \end{bmatrix}$ has index 3 and $\begin{bmatrix} 0 & -12 \\ 10 & 1 \end{bmatrix}$ has index 3 (all are reduced representatives in different similarity classes).

We were not able to prove (or disprove) this Conjecture.

5 Uniquely nil-clean matrices

In this section we describe a (large) class of uniquely nil-clean matrices.

Theorem 6. *For a nonnegative integer n consider the (reduced) matrix* $M_n = \begin{bmatrix} n+1 & -3(n+1) \\ 3n+2 & -n \end{bmatrix}$. *For $n = 3k$ and $n = 3k+2$, M_n is nil-clean of index 2 and for $n = 3k+1$, M_n is uniquely nil-clean (and not strongly nil-clean).*

Proof. Since $M_n = \begin{bmatrix} 2(n+1) & -2(n+1) \\ 2n+1 & -2n-1 \end{bmatrix} + \begin{bmatrix} -n-1 & -n-1 \\ n+1 & n+1 \end{bmatrix}$, all these matrices are nil-clean.

For the matrix M_n, the equation (3) is

$$-3(n+1)x^2 - (2n+1)xy - (3n+2)y^2 - 3(n+1)x - (8n^2+15n+6)y = 0.$$

Multiplying the equation (3) by $-12(n+1)$ we obtain

$[6(n+1)x + (2n+1)y + 3(n+1)]^2 + (32n^2+56n+23)y^2 + (96n^3+264n^2+234n+66)y - 9(n+1)^2 = 0$.

Next we perform the substitution $m = -6(n+1)x - (2n+1)y - 3(n+1)$ which gives: $m^2 + (32n^2+56n+23)y^2 + (96n^3+264n^2+234n+66)y - 9(n+1)^2 = 0$.

Since m^2 is always greater than, or equal to zero,

$$f(n,y) = (32n^2+56n+23)y^2 + (96n^3+264n^2+234n+66)y - 9(n+1)^2$$

must be less than, or equal to zero. This is verified in the segment limited by the roots.

The roots are: $\dfrac{-6(n+1)(16n^2+28n+11) \pm 6(n+1)(16n^2+28n+12)}{2(32n^2+56n+23)}$,

that is, $-3(n+1)$ and $\dfrac{3(n+1)}{32n^2+56n+23}$.

All values of y from $-3(n+1)$ to 0 should be replaced in $f(n,y)$. The result should be the negative (or zero) of a perfect square. Notice that $96n^3+264n^2+234n+66 = 6(n+1)(16n^2+28n+11)$.

Next we show that $f(n,y)$ is the negative (or zero) of a perfect square only if $n+1$ divides y.

Write $y = -(n+1)t$ with real $t \in [0,3]$ and replace into $f(n,y)$. If this is a square of an integer we have to show that $t \in \mathbf{Z}$. Equivalently, $9 + 6(16n^2+28n+11)t - (32n^2+56n+23)t^2$ is a square only if $t \in \mathbf{Z}$ (here $t \in [0,3]$). If $l = (4n+3)(n+1)$, this amounts to $(t-3)^2 - 8lt(3-t)$ is a square. However, for $0 \leq t \leq 3$ (and $l \geq 3$) the only possible integer squares are indeed, $0, 1, 4$ or 9 and so $t \in \mathbf{Z}$.

Hence $f(n,y)$ is the negative (or zero) of a perfect square iff $t \in \{0,2,3\}$ because it equals 3^2 for $t=0$, $(8n+7)^2$ for $t=2$ and 0 for $t=3$.

Therefore, the values of y we have to check are:

1. $-3(n+1)$, which replaced in $f(n,y)$ gives $-9(n+1)^2[-32n^2-56n-23+2(16n^2+28n+11)+1] = 0$. Then $m = -6(n+1)x-(2n+1)y-3(n+1) = 0$ with $y = -3(n+1)$ gives $x = n$, that is, the solution $(a,b) = (n, -3(n+1))$.

2. $-2(n+1)$, which replaced in $f(n,y)$ gives $-[(n+1)(8n+7)]^2$. Then $m = \pm(n+1)(8n+7)$ with $y = -2(n+1)$ give $x = 2n-1$ and $x = -\dfrac{2(n+2)}{3}$. Here we obtain the solution $(2n+1, -2(n+1))$ and, (only) if $n = 3k+1$, also $(-2(k+1), -2(n+1))$.

3. $-(n+1)$, which replaced in $f(n,y)$ gives $-[(8n+7)^2+3]$ and it is not hard to see that $(8n+7)^2+3$ cannot be a perfect square (the difference of two squares is 3 only for 1^2 and 2^2).

4. 0, which replaced in $f(n,y)$ gives $-[3(n+1)]^2$. Here $m = \pm 3(n+1)$ gives $x = 0$ or $x = -1$, i.e. the solutions $(0,0)$ or $(-1,0)$.

Summarizing,

For $n = 3k$, the equation (3) has four solutions: $(0,0)$, $(-1,0)$, $(a,b) = (n, -3(n+1))$ and $(2n+1, -2(n+1))$. The first two are eliminated by Lemma 1, the fourth gives the decomposition already mentioned and the third gives the decomposition $M_n = \begin{bmatrix} n+1 & -3(n+1) \\ k & -n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 8k+2 & 0 \end{bmatrix}$, so index 2.

For $n = 3k+2$, the equation (3) has the same four solutions as in the previous case, but now $(0,0)$ and (a,b) are eliminated by Lemma 1. The remaining solutions give the decomposition above and $M_n = \begin{bmatrix} 0 & 0 \\ 8k+7 & 1 \end{bmatrix} + \begin{bmatrix} n+1 & -3(n+1) \\ k+1 & -n-1 \end{bmatrix}$, so again index 2.

For $n = 3k+1$, the equation (3) has five solutions: the four mentioned above and $(-2(k+1), -2(n+1))$. The first three are eliminated by Lemma 1, and the fourth gives the (unique) nil-clean decomposition mentioned above, since the last solution does not verify (2). Indeed, here (2) is

$$(2n+1)x + (3n+2)y - 3(n+1)z = -(8n^2+15n+6)$$

and so $3(n+1)z = -2(k+1)(2n+1) - 2(n+1)(3n+2) + 8n^2 + 15n + 6 = 3(2k+1)(k+1)$ and so $(3k+1)z = (2k+1)(k+1)$. Since $\gcd(3k+1, 2k+1) = 1$ this amounts to $3k+1$ divides $k+1$, impossible for $k \geq 1$.

These decompositions are not strongly nil-clean (by computation or by Theorem 4). □

By relating the similarity classes of nil-clean matrices to class numbers of quadratic integer rings, a different type of results may be proved. However, in order not to further lengthen this paper, this is done elsewhere.

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