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MORPHIC ABELIAN GROUPS

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An R-module ${}_RM$ is called *morphic* if $M/\operatorname{im}\alpha \cong \ker\alpha$ for every endomorphism α of M, that is, if the dual of the Noether isomorphism theorem holds. Mostly all morphic $\mathbf{Z}$-modules are determined leaving open some classes of nonsplitting mixed groups, which actually cannot be completely characterized. However, for these classes, comprehensive information including negative examples is given.

Keywords: Morphic module; Ulm–Kaplansky invariant; Abelian group; rank; Hopfian; Cohopfian; Dedekind finite.

1. Introduction

In [3], Ehrlich showed that an endomorphism α of a module ${}_RM$ is unit-regular if and only if it is regular and $M/\operatorname{im}\alpha \cong \ker\alpha$. An endomorphism α of a module ${}_RM$ is called *morphic* if $M/\operatorname{im}\alpha \cong \ker\alpha$, that is, if the dual of the Noether isomorphism theorem holds for α. The module ${}_RM$ is called *morphic* if every endomorphism is morphic.

In [9], Nicholson and Campos characterized the finitely generated morphic Abelian groups.

In this paper, the morphic Abelian groups (i.e. $\mathbf{Z}$-modules) are investigated, giving a complete characterization for the torsion-free or divisible, or torsion, or splitting mixed groups. Moreover, fairly good information is given for the remaining nonsplitting case and for the divisible part decomposition. As a general conclusion, morphic Abelian groups are rare phenomenon and they behave badly with respect to (even split) extensions.

A left R-module M is called *Hopfian* if every surjective R-endomorphism is invertible, and *co-Hopfian* if every injective R-endomorphism is invertible. It is *DF* (Dedekind finite) if it is not isomorphic to any proper direct summand of itself (equivalently, $\operatorname{End}_R(M)$, its ring of R-endomorphisms, is Dedekind finite).

In order to give the reader a better perspective, our investigation is, whenever possible, related to the following implications

$$\text{morphic} \Rightarrow \text{Hopfian, co-Hopfian} \Rightarrow \text{DF}.$$

The word *group* will be used to designate an Abelian group G. $T(G)$ is its torsion subgroup, G_p its p-primary component for each prime p, and $f_G(n)$ denotes the nth Ulm–Kaplansky invariant of a p-group G. For unexplained terminology we refer to Fuchs [4].

2. Divisible or Torsion-Free

An immediate consequence of the definition is that *a morphic endomorphism is monic if and only if it is epic.*

From [9], we mention the following simple (but useful) characterization for morphic modules in terms of submodules: *a module M is morphic if and only if for submodules N and H of M*, if $M/N \cong H$ then $M/H \cong N$.

Further, notice that (see [9]) *direct summands of morphic modules are morphic.*

Since an (Abelian) group is torsion-free (or divisible) exactly when all multiplications with positive integers are injective (respectively surjective) and a morphic endomorphism is epic if and only if it is monic, the next simple but (surprising and) important result follows at once.

Proposition 1. (i) *A torsion-free group is morphic only if it is divisible.* (ii) *A divisible group is morphic only if it is torsion-free.*

Therefore, *there are no nonzero torsion (or mixed) divisible morphic groups and the only torsion-free morphic groups are divisible.*

Since direct sums of infinitely many copies of $\mathbf{Q}$ are not even DF, and so nor morphic, the only possible divisible morphic groups are the finite (n copies) direct sums $\mathbf{Q} \oplus \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$.

We first give an elementary proof for $n = 1$.

Lemma 2. $\mathbf{Q}$ *is a morphic* $\mathbf{Z}$*-module.*

Proof. Since every proper factor group of $\mathbf{Q}$ is torsion, $\mathbf{Q}/N \cong H$ where N and H are submodules of $\mathbf{Q}$ is possible only if $N \in \{0, \mathbf{Q}\}$. In both cases, we also have $\mathbf{Q}/H \cong N$. Thus, the statement follows from the characterization cited above. □

Proposition 3. *Any finite direct sum of* $\mathbf{Q}$ (*equivalently, finite rank torsion-free divisible group*) *is morphic.*

Proof. It is well-known that $\text{End}_{\mathbf{Z}}(\mathbf{Q}^n) \cong \mathcal{M}_n(\mathbf{Q})$ which is semisimple and so unit-regular. It remains to use Example 28, [9]: modules with unit-regular endomorphism rings are morphic. □

Theorem 4. *The following conditions are equivalent*:

(a) *G is torsion-free morphic*;
(b) *G is divisible morphic*;
(c) $G \cong \mathbf{Q} \oplus \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$, *i.e. a finite direct sum of* $\mathbf{Q}$.

Remarks. (1) Notice that morphic torsion-free (or divisible) groups are countable. (2) It is easy to see that *a torsion-free group which is not divisible is not Hopfian.*

Indeed if G is not divisible, there exists a nonzero integer n and $g \in G$ such that $nx = g$ has no solution in G. Then the multiplication with n is an isomorphism of G on a proper subgroup.

3. Torsion

Since there are no morphic torsion divisible groups, the morphic torsion groups are reduced.

First an expected reduction is the following proposition.

Proposition 5. *A torsion group is morphic if and only if all its primary components are morphic.*

Proof. The primary components being direct summands, the conditions are necessary.

Conversely, let $G = \bigoplus G_p$ be the decomposition of G into p-components. Each subgroup N of G has a corresponding decomposition into primary components $N = \bigoplus N_p$ with N_p subgroup of G_p, for all p. Therefore, $G/N \cong \bigoplus (G_p/N_p)$.

Now suppose $G/N \cong H$ for a subgroup H of G. Using a similar decomposition of $H = \bigoplus H_p$, $\bigoplus (G_p/N_p) \cong \bigoplus H_p$ and so $G_p/N_p \cong H_p$. Since every G_p is morphic, $G_p/H_p \cong N_p$ and accordingly $G/H \cong N$, as desired. □

Therefore, only the *morphic p-groups* must be singled out.

First we mention the following lemma.

Lemma 6. *If $m < n$ are positive integers, the direct sum $A = \mathbf{Z}_{p^m} \oplus \mathbf{Z}_{p^n}$ is not morphic.*

Proof. Consider $N = \mathbf{Z}_{p^m} \oplus p^m\mathbf{Z}_{p^n}$ and $H = \mathbf{Z}_{p^m} \oplus 0$. Then $A/N \cong \mathbf{Z}_{p^m} \cong H$ but $A/H \cong \mathbf{Z}_{p^n} \ncong N$. □

Recall that a direct sum of cyclic p-groups is called *homogeneous* if all its direct summands are isomorphic.

Theorem 7. *A (reduced) p-group G is morphic if and only if it is finite and homogeneous.*

Proof. Since the condition is sufficient according to [9, Theorem 26] (such p-groups are finitely generated), suppose G is a morphic p-group and let B be a basic subgroup of G.

We first show that B must be finite and homogeneous, that is, all its Ulm–Kaplanky invariants are zero, with (at most) one exception, say $f_G(n)$, which is finite. Indeed, if two different invariants are not zero, say $f_G(m) \neq 0 \neq f_G(n)$ with $m < n$ then the basic subgroup contains a direct summand $\mathbf{Z}_{p^m} \oplus \mathbf{Z}_{p^n}$, which is pure and bounded in G, so a direct summand of G. But this contradicts the previous Lemma.

Further on, if the only nonzero invariant is infinite, arguing as above, G contains a direct summand which consists in a direct sum of infinitely many isomorphic cyclic p-groups, which is not morphic (not even DF), so again a contradiction.

Finally, if G contains a finite homogeneous basic subgroup, again as pure and bounded, this must be a direct summand of G, say $G \cong B \oplus (G/B)$ where G/B is divisible. Since G/B is morphic as a direct summand of G, we must have $G/B = 0$ (there are no morphic divisible p-groups) and so $G = B$ is finite and homogeneous. □

The reader may compare this with

Theorem (2.9) [2]. *A reduced p-group is not DF if and only if there is an $n \geq 0$ with infinite $f_G(n)$ (the nth Ulm invariant), or, rephrasing, a reduced p-group is DF if and only if all its Ulm–Kaplansky invariants are finite.*

Notice that morphic torsion groups are also countable.

4. Mixed Groups: Positive Results and Negative Examples

In [8], the following two questions were stated: When is the direct sum ${}_RM \oplus {}_RN$ morphic?, with a special case, when is ${}_RM \oplus {}_RM$ morphic?

Since these questions are related to some "traditional' reduction theorems in Abelian Group Theory (i.e. for the divisible part $D(G)$, and a reduced group R, both morphic, when is $D(G) \oplus R$ also morphic, and, for the torsion part $T(G)$ and a torsion-free group F, both morphic, when is $T(G) \oplus F$ also morphic), next we gather some simple conclusions, consequences of the previous sections.

Proposition 8. *For a group G, the direct sum $G \oplus G$ is morphic in the following cases*: (i) *G is any torsion morphic group*; (ii) *G is any torsion-free morphic group.*

This remains true for any direct sum of finitely many copies of G.

For the next two results we need (see [9])

Lemma 9. *If ${}_RM$ and ${}_RN$ are morphic modules for which* $\mathrm{Hom}_R(M, N) = 0 = \mathrm{Hom}_R(N, M)$, *then $M \oplus N$ is morphic.*

Proposition 10. *Let $G = D(G) \oplus R$ be the decomposition of a group G with the divisible part $D(G)$. If both $D(G)$ and R are morphic and R is torsion, then G is* (*splitting and*) *morphic.*

Proof. In this case $G = D(G) \oplus T(G)$, so we use $\mathrm{Hom}_{\mathbf{Z}}(\text{divisible}, \text{reduced}) = 0 = \mathrm{Hom}_{\mathbf{Z}}(\text{torsion}, \text{torsion-free})$ in the previous lemma. □

Proposition 11. *If both $T(G)$ and $G/T(G)$ are morphic and $T(G)$ has only finitely many p-components then G is (splitting and) morphic.*

Proof. If $T(G)$ is morphic and has only finitely many primary components it is finite and so G splits (by Baer–Fomin celebrated theorem). If $G/T(G)$ is morphic, it is both torsion-free and divisible and once again we use Lemma 9. □

Actually, splitting mixed morphic groups are characterized by the following theorem.

Theorem 12. *The splitting morphic mixed groups are exactly the groups $G = T(G) \oplus D(G) = \bigoplus_p (\mathbf{Z}(p^{k_p})^{n_p} \oplus \mathbf{Q}^k$ with nonnegative integers k_p, n_p, and k.*

Proof. Indeed, by our previous results, any splitting morphic group must be such a direct sum. Conversely, a direct sum of this type, satisfies $T(G)$ morphic, $G/T(G) = D(G)$ morphic, and is morphic once again because of Lemma 9. □

Remark 13. These groups are exactly those whose endomorphism ring is left (but not right) self-injective (see [10] or [6]).

Further on, we give some necessary conditions for a group to be morphic, in the general (nonsplitting) case.

First we prove an elementary

Lemma 14. *For any morphic group G, the elementary p-groups G/pG and $G[p]$ have the same rank if and only if $pG + T(G) = G$.*

Proof. In order to avoid the usual writing complications, we just mention here a finite-dimensional proof.

Using a Noether isomorphism theorem

$$\frac{G/pG}{(pG + T(G))/pG} \cong \frac{G}{pG + T(G)},$$

since all ranks are dimensions of vector spaces over $\mathbf{Z}_p$, the claim is obvious for G/pG and $(pG + T(G))/pG$. Finally, another Noether isomorphism gives $(pG + T(G))/pG \cong T(G)/pG \cap T(G) = T(G)/pT(G)$, since the torsion part is pure, and the dimensions $r(T(G)/pT(G)) = r(G[p])$ (indeed, both give the p-rank of the group G, if the p-components are homogeneous). □

Proposition 15. *If G is morphic then the torsion part $T(G)$ is morphic and $G/T(G)$ is divisible.*

Proof. As already noticed $T(G)$ is always reduced. If $T(G)$ is not morphic then at least one primary component is not morphic. According to Theorem 7, a primary component is not finite or not homogeneous. As in the proof of the theorem, one can show that G is not morphic.

As for the second claim, suppose $G/T(G)$ is not divisible. Then there is a prime number p such that $p(G/T(G)) \neq G/T(G)$, or equivalently, $pG + T(G) \neq \dot{G}$. According to the previous lemma, the elementary p-groups G/pG and $G[p]$ are not isomorphic. Hence, the multiplication with p in G (denoted μ_p) is not morphic (indeed, $\ker \mu_p = G[p]$ and $\operatorname{im} \mu_p = pG$). □

For morphic reduced (mixed) groups it is worth mentioning a celebrated environment, a class of groups which was under close scrutiny the last 15 years, for Abelian group theorists. In [5], a class of reduced mixed groups of finite torsion-free rank, denoted Γ was defined for the study of regular or PP (principal projective) endomorphism rings of mixed (Abelian) groups, as follows: $G \in \Gamma$ if there is a pure embedding $\bigoplus G_p < G < \prod G_p$.

Then it can be proved

Lemma 16. *A reduced (mixed) group G of finite torsion-free rank belongs to Γ if and only if for all primes p, the p-component is a direct summand of G, and, $G/T(G)$ is divisible.*

Therefore, using our previous results (every p-component of a morphic group is pure and bounded, so a direct summand) we obtain at once the following proposition.

Proposition 17. *Every morphic reduced (mixed) group G of finite torsion-free rank belongs to Γ.*

The interest for reduced (mixed) groups H, with divisible $H/T(H)$ and (only) homogeneous p-components is not new. In the 1960s (see Rangaswamy [10]), for such groups, and so *for reduced morphic groups*, the following seven properties were already noticed:

(a) every subgroup of a H_p is an endomorphic image of H;
(b) H is Hausdorff in the n-adic topology;
(c) S is pure in H if and only if $S/T(S)$ is divisible and $T(S)$ is a direct summand in $T(H)$;
(d) if $S = \operatorname{im} \alpha$ for $\alpha \in \operatorname{End}(H)$ then $T(S) \neq 0$ and $S/T(S)$ is divisible;
(e) the n-adic closure of a pure subgroup is pure;
(f) H has no closed torsion-free subgroups;
(g) if $\alpha \in \operatorname{End}(H)$ then $\operatorname{im} \alpha$ is pure in H if and only if $\ker \alpha$ is pure in H.

As in the Hopfian, co-Hopfian and DF cases, $G/T(G)$ may not be morphic, even if G is morphic.

Example 18. The group $G = \prod_p \mathbf{Z}(p)$ is morphic, not countable, nor splitting, but $G/T(G)$ is not morphic.

Indeed, as direct product $\prod_p \mathbf{Z}_p$ of fields, the endomorphism ring of G is commutative, so G is DF, but $G/T(G)$ is not DF (nor morphic: it is infinite rank torsion-free divisible). Moreover, since every field is unit-regular, and a direct product of rings is unit-regular if and only if each factor is unit-regular, G is morphic because $\mathrm{End}(G)$ is unit-regular (see Example 28 [9]).

Further, we show that the usual expected reconstruction, "*if both $D(G)$ and R are morphic then $G = D(G) \oplus R$ is morphic*", fails.

We first recall from [9], as a special case of Lemma 24, the following useful lemma.

Lemma 19. *If the direct sum of R-modules $N \oplus K$ is morphic and there exists a R-linear epimorphism $\lambda : K \to N$ then $K \cong N \oplus \ker \lambda$.*

Obviously, this can be repharased as: *if there exists a R-linear epimorphism $\lambda : K \to N$ and K has no direct summands isomorphic to N then $N \oplus K$ is not morphic.*

Therefore, we obtain at once the following corollary.

Corollary 20. *Let G be a reduced (mixed) group. If there is a surjective group homomorphism $\pi : G \to \mathbf{Q}$ then $M = \mathbf{Q} \oplus G$ is not morphic.*

Remark 21. Since $\mathbf{Q}$ is countably generated, for the existence of an epimorphism $G \to \mathbf{Q}$ we just need the torsion-free rank $r_0(G) \geq \aleph_0$.

Example 22. For $M = \mathbf{Q} \oplus G$, with G from the previous example, it is known that $\mathrm{End}(M)$ is 2-regular but not regular (and so, nor unit-regular). Moreover, M is not morphic.

Indeed, we just take $\pi : G \to G/T(G) \to \mathbf{Q}$ ($G/T(G)$ is torsion-free divisible), the composition of two projections, in the previous corollary.

Recall (see [2]) that if N is a fully invariant subgroup of G and both N, G/N are DF then G is DF, and, a similar property was proved for Hopfian groups by Baer ([1] but only in the special case $N = T(G)$).

Finally, we show that the corresponding usual expected reconstruction, "*if both $T(G)$ and $G/T(G)$ are morphic then G is morphic*", fails too.

Example 23. Let $H = \mathbf{Q} \oplus P$ with the subgroup $P = P(G, a) = \{g \in G : ng \in \langle a \rangle$ for some positive integer $n\}$ of all elements in G that depend on $\{a\}$ (here once again $G = \prod_p \mathbf{Z}(p)$ and a is the infinite order element $(\overline{1}, \overline{1}, \ldots)$). Then $T(H)$ and $H/T(H)$ are both morphic, but H is not morphic.

Indeed, it is known that P is the smallest pure subgroup which includes $T(G) = \bigoplus_p \mathbf{Z}(p)$ and a. Moreover, P has torsion-free rank 1, $T(P) = T(G)$ and $P/T(P) \cong \mathbf{Q}$. Then $T(H) = T(P) = T(G) = \bigoplus_p \mathbf{Z}(p)$ is morphic (according to our Sec. 3), and $H/T(H) \cong \mathbf{Q} \oplus \mathbf{Q}$ is also morphic (see Sec. 2). Once again we take $f \in \mathrm{End}(H)$ given by $\begin{bmatrix} 0 & \pi \\ 0 & 1_P \end{bmatrix}$, with the projection $\pi : P \to P/T(P)$. Hence H is not morphic according to the previous corollary.

Question. $D = \mathbf{Q}^n$ and the existence of an epimorphism $\pi : R \to D$ are both necessary conditions for $D \oplus R$ to be morphic. Are these two conditions also sufficient?

5. Final Remarks and Open Problems

(1) One can consider morphic Abelian groups in two somehow different ways: Abelian groups which are morphic as **Z**-modules, respectively, Abelian groups whose endomorphism rings are left (respectively right) morphic. The second version is addressed elsewhere.

(2) Comparing morphic modules with DF modules shows that DF is simpler to handle because M is DF if and only if $\mathrm{End}_R(M)$ is DF, a well-known (and simple) ring theoretic property.

But a ring was called *left morphic* if ${}_RR$ is morphic (as left module), so that (for this choice of definitions), we do not have M is morphic if and only if $\mathrm{End}_R(M)$ is left morphic.

Therefore, this study would be simplified, if we could

Question: find a *ring theoretic property* $\mathcal{MO}$ (maybe even *ER-property* — see [7]) such that M is morphic if and only if $\mathrm{End}_R(M)$ is $\mathcal{MO}$.

(3) **Question:** Determine the rings R, such that a module is morphic if and only if it is Hopfian and co-Hopfian.

For $R = \mathbf{Z}$ this fails: indeed, every finite group is Hopfian and co-Hopfian but, as we already saw, $\mathbf{Z}(2) \oplus \mathbf{Z}(4)$ is not morphic.

(4) The morphic definition can be reconsidered from another point of view: a submodule N of an R-module M *is* (say) *relatively morphic* if $M/K \cong N$ whenever $M/N \cong K$.

Question (considered but not addressed by the author as early as year 2000): Find the submodules of a given module which are relatively morphic (e.g. the zero submodule N has this property in M if and only if M is co-Hopfian); find the modules in which every submodule is relatively morphic.

(5) Lemma 1 (2) [9], that is: an endomorphism α is morphic if and only if there is an endomorphism β such that $\mathrm{im}\,\beta = \ker \alpha$ and $\mathrm{im}\,\alpha = \ker \beta$, indicates that the definition of morphic endomorphism is categorical. Therefore the following is legitimate.

Definition 24. Let M be an object in an exact additive category $\mathcal{C}$. An endomorphism α of M is called *morphic* if there is an endomorphism $\beta \in \mathrm{End}_{\mathcal{C}}(M)$ such that the following sequence is exact

$$M \xrightarrow{\alpha} M \xrightarrow{\beta} M \xrightarrow{\alpha} M.$$

The object M is itself *morphic* if every $\alpha \in \mathrm{End}_{\mathcal{C}}(M)$ is morphic.

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