On type 1 representable lattices of dimension at most 4

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Abstract. We prove that every lattice of length (formerly, *dimension*) at most 4, representable by commuting equivalence relations, is also representable by subgroups of an Abelian group.

1. Introduction

In [5], Jónsson considered lattices of commuting equivalence relations (often called *type* 1 *representable*) and proved that such lattices are Arguesian. In [6], Jónsson was able to prove that for lattices of dimension (from now on *length*, to use the current terminology) at most 4, the converse holds true.

After 35 years, the paper [3] made it possible to add $n = 5$ to the former $n \leq 4$: every Arguesian lattice of length at most 5 is type 1 representable.

In the reversed direction, examples given by Haiman [4], show that for $n \geq 7$, there are Arguesian lattices which are not type 1 representable.

Denote by $\mathcal{A}$ the class of all the lattices isomorphic to subgroup lattices of Abelian groups, by $\mathcal{N}$ the class of all the lattices isomorphic to normal subgroup lattices of arbitrary groups and

by $\mathcal{L}$ the class of all the linear lattices (i.e., isomorphic to lattices of commuting equivalence relations). Further, denote by $\mathbf{Z}$ the ring of all integers, and by $\mathbf{Z}_n$ the ring of integers modulo a positive integer n. Furthermore, denote by $\mathcal{L}(\mathbf{Z})$ the class of all the lattices representable by lattices in $\mathcal{A}$, by $\mathcal{N}(\text{rep})$ the class of all lattices representable by lattices in $\mathcal{N}$, and by $\mathcal{T}_1$ the class of all the type 1 representable lattices, that is, representable by lattices in $\mathcal{L}$. For general information on representations of lattices we refer the reader to Crawley, Dilworth (see [2], chapter 12, p. 96–104), Nation (see [8], chapter 4, p. 35–43) and to Jónsson's seminal paper (see [5]).

Since all subgroups in an Abelian group are normal and the congruences induced by normal subgroups commute, it is well-known that $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{L}$ and so $\mathcal{L}(\mathbf{Z}) \subseteq \mathcal{N}(\text{rep}) \subseteq \mathcal{T}_1$, and that none of these inclusions is an equality (for counterexamples see our final comment, Jónsson [7, p. 309–314], and

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Pálfy–Szabó [9]). All lattices in these classes are Arguesian and hence also modular. Combining results of Birkhoff, Frink, Schützenberger and Jónsson, if we call *geomodular* a complete, atomistic, complemented, modular lattice such that every atom is compact, the following result was already known in the early fifties:

Theorem 1.1. *The following conditions are equivalent, for any geomodular lattice L:*

(i) $L \in \mathcal{A}$*;*
(ii) $L \in \mathcal{N}$*;*
(iii) $L \in \mathcal{L}$*;*
(iv) L *is Arguesian.*

In [7], Jónsson was able to extend the previous theorem *for any complemented* (*modular*) *lattice*. In the same paper, Jónsson gives an example of a modular lattice of length 5 which is isomorphic to a lattice of commuting equivalence relations, but not to any lattice of normal subgroups of an arbitrary group. It follows that $\mathcal{N} \subsetneq \mathcal{L}$, with a (modular) lattice of length 5 in the difference.

Finally, in [2], the following result is proved.

Theorem 1.2. *The following conditions are equivalent, for any complemented modular lattice L:*

(a) $L \in \mathcal{L}(\mathbf{Z})$*;*
(b) $L \in \mathcal{T}_1$*;*
(c) L *is Arguesian.*

We quote from [5]: "It can be shown that a lattice of dimension 4 or less has a type 1 representation if and only if it is Arguesian. Similar questions can be raised

concerning lattices which are isomorphic to lattices of normal subgroups of arbitrary groups or to lattices of subgroups of Abelian groups".

Some 60 years after this comment, in this paper we show that we have equalities $\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1$ for length at most 4 (modular) lattices.

This study can also be related to the following (frequently hard) open question: when is a given quasivariety, actually a variety? Actually, here the closure under homomorphic images makes the difference.

For the classes $\mathcal{L}(\mathbf{Z})$, $\mathcal{N}(\text{rep})$, and $\mathcal{T}_1$, the answer is not known. Since our study shows that for length at most 4 all these classes coincide with the Arguesian lattices and Arguesian lattices form a variety (i.e., it is closed under homomorphic images), our results somehow encourage to conjecture a positive answer.

2. Modular lattices of length at most 4

In [6], a "reasonably complete picture" of all modular lattices of length at most 4 is given. Since this is also our environment (as it was for Arguesian lattices), we briefly remind the reader the results.

A *lattice of length zero* consists of just one element $0 = 1$, and a *lattice of length one* consists of exactly one chain with two

elements 0 and 1. A *lattice of length two* consists of 0 and 1 together with one or more atoms and will be denoted by $\mathcal{M}_n$, if it has n atoms. Since the join of two distinct atoms is always 1, and the meet is always 0, such a lattice is completely determined up to isomorphism by the number of its atoms.

Remarks 2.1. – If L and L' are *lattices of length two* and L' has at least as many elements as L, then L is isomorphic to a sublattice of L'. In fact, given atoms p in L and p' in L', there exists an isomorphism f from L to L' such that $f(p) = p'$.

– All these lattices are complemented, so the equalities $\mathcal{A} = \mathcal{N} = \mathcal{L}$ and $\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\mathrm{rep}) = \mathcal{T}_1$ hold according to Theorems 0 and $0'$ (see previous Section), respectively. Thus, we can discard at once all lattices of length at most 2.

Denote by $\delta(x)$ the height of an element x in a modular lattice L of finite length, by s the *socle of* L (the join of all atoms) and by r the *radical of* L (the meet of all dual atoms). Since *a finite length modular lattice is complemented if and only if* 1 *is the socle if and only if* 0 *is the radical*, the conditions $\delta(s) = n$ and $\delta(r) = 0$ are equivalent and imply that L is complemented.

If $\delta(s) = 1$, then s is an atom of L, and in fact s is the only atom of L. In this case L is completely determined by its sublattice $1/s$ of length $n - 1$. Similarly, if $\delta(r) = n - 1$, then the study of L reduces to the study of its length $n - 1$ sublattice $r/0$. We shall therefore be concerned here with the cases $1 < \delta(s) < n$ and $0 < \delta(r) < n - 1$.

Thus if $n = 3$ then only $\delta(s) = 2$ and $\delta(r) = 1$ has to be considered and if $n = 4$, $\delta(s) \in \{2, 3\}$ and $\delta(r) \in \{1, 2\}$.

Say that an element $x \in L - \{1\}$ is *meet-irreducible* if $x = u \wedge v$ implies that $x \in \{u, v\}$, for all $u, v \in L$; *join-irreducible elements* are defined dually. An element is *irreducible* if it is both meet- and join-irreducible.

Every join-irreducible element covers a unique element, and every meet-irreducible element is covered by a unique element. Thus, in the context above, if s is not an atom (e.g., if $\delta(s) = 2$), then it is not join-irreducible, and (dually) if r is not a dual atom (e.g., if $\delta(r) = 2$), then it is not meet-irreducible.

Therefore we need to distinguish only the following two cases.

Theorem 2.2 (Theorem 2.2 of [6])**.** *For $n \in \{3, 4\}$, if $0 < \delta(r) < \delta(s) < n$, then $r < s$ and $L = s/0 \cup 1/r$.*

Theorem 2.3 (Theorem 2.3 of [6]). *If $n = 4$ and $\delta(r) = \delta(s) = 2$, then $s/0 \cup 1/r = L - X$, where X is the set of all irreducible elements $x \in L$ with $\delta(x) = 2$. Furthermore, each element of X covers a unique atom and is covered by a unique dual atom, and two elements of X cover the same atom if and only if they are covered by the same dual atom. Finally, if $s \neq r$, then $s \wedge r$ is an atom and is covered by r, $s \vee r$ is a dual atom and covers s, and $s \wedge r \prec x \prec s \vee r$ for every element $x \in X$.*

3. Length 3 and 4

A modular non complemented lattice of length 3 is isomorphic (see previous Section) to a lattice as on Figure 1, with s an atom and r not a dual atom. This diagram represents the lattice $\mathcal{M}_n$ glued with the lattice $\mathcal{M}_m$, with a prime ideal of the top lattice being identified with a prime filter of the bottom lattice. We will denote this lattice by $\mathcal{M}_n/\mathcal{M}_m$.

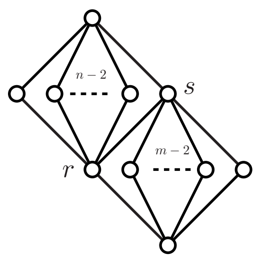

FIGURE 1. A family of lattices of length 3

Since finite Abelian groups are self-dual (see [1] Baer, 1937), if $m \neq n$, then clearly $\mathcal{M}_n/\mathcal{M}_m \notin \mathcal{A}$. Moreover, it can be proved that only $\mathcal{M}_{p+1}/\mathcal{M}_{p+1} = L(\mathbf{Z}_p \oplus \mathbf{Z}_{p^2}) \in \mathcal{A}$, for a prime number p.

Thus we obtain the following.

Theorem 3.1. *Every modular non complemented lattice of length* 3 *belongs to $\mathcal{L}(\mathbf{Z})$.*

Proof. For arbitrary given positive integers m, n, choose any prime number p such that $\max(m, n) \leq p + 1$. A lattice embedding must be defined from $\mathcal{M}_n/\mathcal{M}_m$ into $\mathcal{M}_{p+1}/\mathcal{M}_{p+1}$; this is illustrated on Figure 2.

This is done as follows: 0, r, b, s, a, 1 remain fixed, the remaining $m - 2$ atoms are injectively mapped to atoms, and the remaining $n - 2$ dual atoms are injectively mapped to dual atoms.

The length 3 case in which either s is an atom or r is a dual atom reduces to the length 2 case: embed, for instance, as shown on Figure 3, for a prime number $p \geq n - 1$.

□

Corollary 3.2. *For lattices of length ≤ 3, $\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1$ holds.*

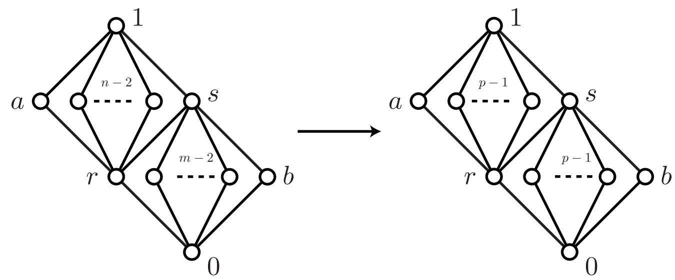

Figure 2. An embedding between lattices of length 3

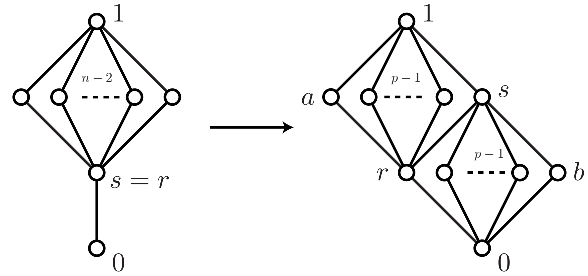

Figure 3. An embedding between lattices of length 3

To conclude our study, a length 4 modular non complemented lattice is isomorphic (see previous Section) either to one of the four types of lattices represented on Figure 4, or, in the case $s = r$, to a lattice of the form represented on Figure 5.

Notice that these Figures improve Jónsson's illustration in [6], p. 138, where in particular Figure 5 is twofold incomplete: the right part did not enter in the page, and some more $\mathcal{M}_n$'s must be added in order to cover all possibilities (for instance $L(\mathbf{Z}_4 \oplus \mathbf{Z}_4)$, see Figure 7).

Using again the fact that for Abelian finite groups, the subgroup lattice is self-dual, one checks that most of the lattices above do not belong to $\mathcal{A}$.

Further, we get the following.

Theorem 3.3. *Every modular non complemented lattice of length* 4 *belongs to* $\mathcal{L}(\mathbf{Z})$.

Proof. First, it is readily seen that the lattice represented in the upper left corner of Figure 4 can be embedded into $L(\mathbf{Z}_p \oplus \mathbf{Z}_{p^3})$, for suitable prime number p. The simplest example of such a lattice in $\mathcal{A}$ is $L(\mathbf{Z}_2 \oplus \mathbf{Z}_8)$, represented in Figure 6.

Moreover, the lattice of Figure 5 can be embedded into $L(\mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2})$, for a suitable prime number p. Again the simplest example in $\mathcal{A}$ is $L(\mathbf{Z}_4 \oplus \mathbf{Z}_4)$, represented in Figure 7.

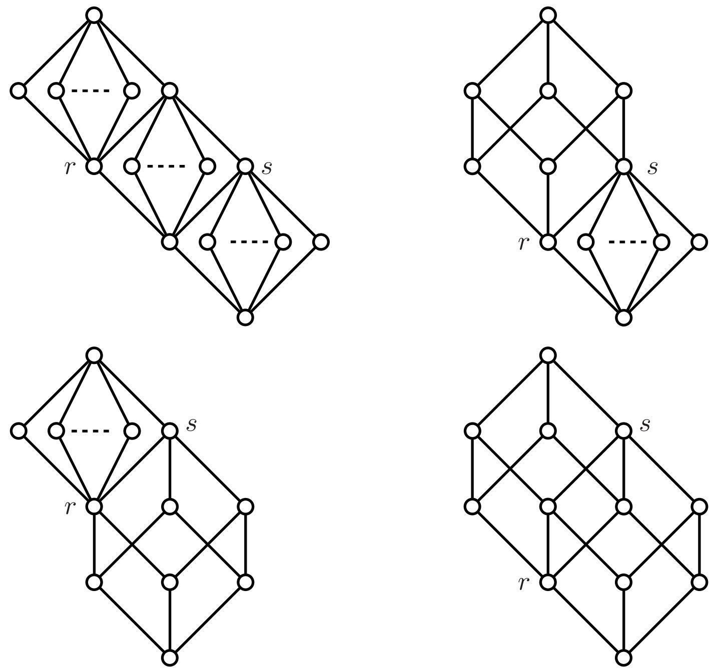

FIGURE 4. Families of lattices of length 4 with $r \neq s$

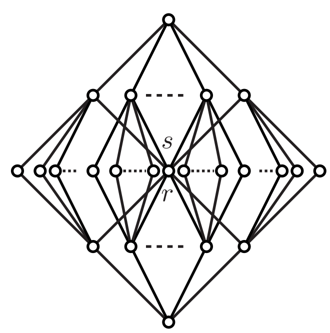

FIGURE 5. A family of lattices of length 4 with $r = s$

For the remaining three types of lattices we deal with a 'cube' (i.e., the direct product $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$, the boolean algebra of length 3, with $\mathbf{2}$ denoting the chain of length 1) over a 'cube', or a 'cube' over or below an $\mathcal{M}_n$.

From now on, we will denote by $\mathbf{3}$ the chain of length 2. Recall that the groups G_i ($i \in I$) are said to be *coprime* if every G_i is a torsion group and $\gcd(\mathrm{ord}(x), \mathrm{ord}(y)) = 1$ for all $x \in G_i$, $y \in G_j$, with $i \neq j$. In the sequel we also use the well-known fact: subgroup lattices of coprime groups commute with direct products.

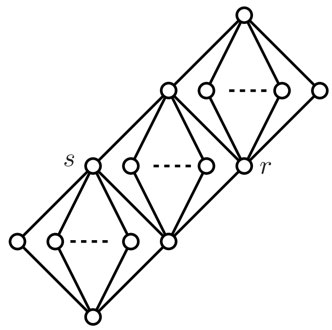

FIGURE 6. A lattice in $\mathcal{A}$, with $r \neq s$

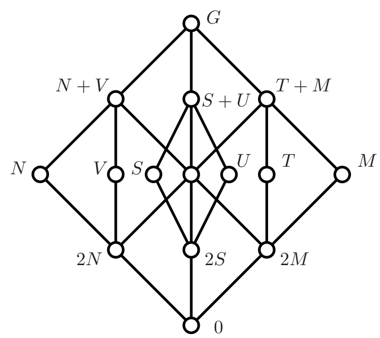

FIGURE 7. A lattice in $\mathcal{A}$, with $r = s$

The lattice represented in the lower right corner of Figure 4 is $L(\mathbf{Z}_{p^2} \oplus \mathbf{Z}_q \oplus \mathbf{Z}_r)$, where p, q, r are pairwise distinct prime numbers. Since $\mathbf{Z}_{p^2}$, $\mathbf{Z}_q$, and $\mathbf{Z}_r$ are coprime, we obtain $L(\mathbf{Z}_{p^2} \oplus \mathbf{Z}_q \oplus \mathbf{Z}_r) \cong L(\mathbf{Z}_{p^2}) \times L(\mathbf{Z}_q) \times L(\mathbf{Z}_r) \cong \mathbf{3} \times \mathbf{2} \times \mathbf{2}$.

We have previously noted that

$$L(\mathbf{Z}_{p^2} \oplus \mathbf{Z}_p) \cong \mathcal{M}_{p+1} \diagup \mathcal{M}_{p+1}.$$

For any distinct prime numbers p and q, $\mathbf{Z}_{p^2} \oplus \mathbf{Z}_p$ and $\mathbf{Z}_q$ are coprime, and so $L(\mathbf{Z}_p \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_q) \cong L(\mathbf{Z}_{p^2} \oplus \mathbf{Z}_p) \times L(\mathbf{Z}_q) \cong (\mathcal{M}_{p+1} \diagup \mathcal{M}_{p+1}) \times \mathbf{2}$. The lattice $L(\mathbf{Z}_2 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_p)$, where p is a prime number and $p > 2$, is represented in Figure 8.

Using this, the lattices represented in the upper right and lower left corner of Figure 4 can be embedded into $(\mathcal{M}_{p+1} \diagup \mathcal{M}_{p+1}) \times \mathbf{2}$, for a suitable prime number p.

Finally, if either s is an atom or r is a dual atom, our study reduces to the length 3 case. We are now concerned with (for instance) the lattices represented in Figure 9.

As for the upper right corner of Figure 4, such a lattice can be embedded into $L(\mathbf{Z}_p \oplus \mathbf{Z}_{p^3})$, for a suitable prime number p. $\square$

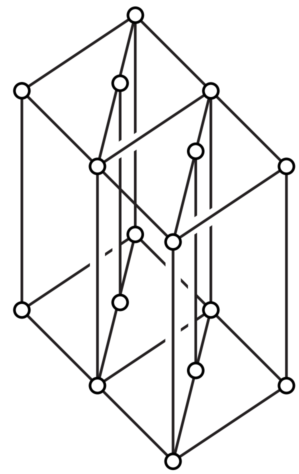

FIGURE 8. Representing $L(\mathbf{Z}_2 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_p)$, for prime $p > 2$

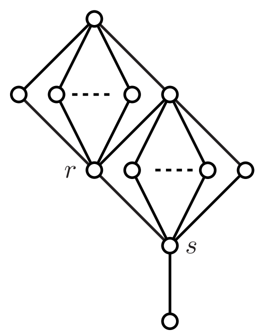

FIGURE 9. A family of lattices of length 4

Corollary 3.4. *For a lattice L of length at most 4, the following properties are equivalent:*

(a) *L is representable by Abelian groups;*
(b) *L is representable by lattices of normal subgroups of arbitrary groups;*
(c) *L is representable by linear lattices;*
(d) *L is Arguesian.*

Remark 3.5. The equality $\mathcal{A} = \mathcal{N}$ does not hold for lattices of length at most 4.

The subgroup lattice of the (8 element) quaternion group, which has length 3, is a simple counterexample.

Whether the equality $\mathcal{N} = \mathcal{L}$ holds for (modular) lattices of length at most 4, remains an open question (for a negative answer, one should "improve" the length 5 example of Jónsson, mentioned in the Introduction).

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