On unit-regular rings.

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Abstract

Among regular rings, unit-regular rings are characterized by complementary idempotents of isomorphic idempotents being also isomorphic. We give a ring theoretical proof of this result (the only existing proof uses internal cancellation for modules). Moreover, we improve (and simplify) another characterization of unit-regular rings given by Camillo and Khurana in [1].

1 Introduction

A ring with identity R is called *unit-regular* if for every element $a \in R$ there is a unit u with $a = aua$.

In regular endomorphism rings of modules, unit-regularity was characterized in the 70's (Ehrlich-Handelman), by a property called *internal cancellation*. Using this, they were able to obtain the following characterization: *let R be a regular ring. R is unit-regular if and only if for every idempotents $e, e' \in R$, $e \cong e'$ implies $1 - e \cong 1 - e'$.*

Since this is the only existing proof (via modules), in this short note, we supply a (direct) ring theoretical proof.

Notice that something similar was done by W. Murray, T. Y. Lam in [4] (1997), where a direct proof for "*a corner ring of a unit-regular ring is also unit-regular*" was given.

Moreover, we improve and simplify another characterization of unit-regular rings, given by V.P. Camillo and D. Khurana (see [1]).

Rings are associative with identity and, for an idempotent e, $\overline{e} = 1 - e$ denotes the complementary idempotent. The set of all the units of a ring R is denoted by $U(R)$.

2 Isomorphic idempotents

Definition 1 *Two idempotents e, e' in a ring R are isomorphic if $eR \cong e'R$ as right R-modules.*

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Equivalently, two idempotents are isomorphic, $e \cong e'$ if and only if there exist $a, b \in R$ such that $e = ab$ and $e' = ba$.

The key result is the following

Lemma 2 *Let e and e' be isomorphic idempotents in a ring R and $e = ab$, $e' = ba$ for elements $a, b \in R$. If $bab = (bab)u(bab)$ holds for a unit $u \in U(R)$ and $c = (1-ue'b)u(1-e')$, $d = (1-e')u^{-1}(1-e)$ then $cd = 1-e$ and $dc = 1-e'$.*

Proof. By left and right multiplication with a, from $bab = (bab)u(bab)$, we obtain $ab = abubab$ and $aba = abuba$. From the first we derive $e(1-ue'b) = 0$ or $(1-e)(1-ue'b) = 1-ue'b$. From the second we deduce $(1-ue'b)ue' = uba - ubabuba = uba - ubaba = uba - uba = 0$.

Thus
$cd = (1-ue'b)u(1-e')u^{-1}(1-e) = (1-ue'b)[1-e-ue'u^{-1}(1-e)] =$
$= 1-e-u\underline{e'b}(1-e) - \underline{(1-ue'b)ue'}u^{-1}(1-e) = 1-e-0-0 =$
$= 1-e$ because $e'b = be$.
Finally,
$dc = (1-e')u^{-1}\overline{(1-e)(1-ue'b)}u(1-e') = (1-e')u^{-1}(1-ue'b)u(1-e') =$
$= (1-e')u^{-1}[\overline{(1-ue'b)u-}\underline{(1-ue'b)ue'}] = (1-e')u^{-1}(1-ue'b)u =$
$1-e'-e'bu+e'bu = 1-e'$. ∎

Having proved this, we can now give a ring theoretical proof for

Theorem 3 *(Ehrlich, Handelmann) A regular ring R is unit-regular if and only if for every two idempotents, $e \cong e'$ implies $1-e \cong 1-e'$.*

Proof. If $e \cong e'$, there are elements $a, b \in R$ with $e = ab$, $e' = ba$. Choose $u \in U(R)$, c and d as in the previous Lemma. Then $cd = 1-e$ and $dc = 1-e'$ and so $1-e \cong 1-e'$. Conversely, let $a \in R$ be an arbitrary element. Since the ring is supposed to be regular, there is an element $x \in R$ such that $a = axa$. Without restriction of generality, we can assume that also $xax = x$. Clearly, ax and xa are isomorphic idempotents in R. Hence there exist elements $c, d \in R$ such that $1-ax = cd$ and $1-xa = dc$. By left and right multiplication with a and x, respectively, we obtain $cda = 0 = adc$ and $xcd = 0 = dcx$. Now consider $u = x + dcd$ and $v = a + cdc$. It is readily checked (notice that both cd and dc are idempotents) that $a = aua$ and $uv = 1 = vu$, that is, $u \in U(R)$, as desired. ∎

In [1] (2001), V.P. Camillo and D. Khurana proved the following,

Theorem 4 *A ring R is unit regular if and only if for every $a \in R$ there is a unit $u \in U(R)$ and an idempotent e such that $a = e + u$ and $aR \cap eR = 0$.*

While trying to give a ring theoretical proof which shows these conditions are necessary (the existing proof uses again internal cancellation for modules), it turned out that this characterization can be improved (simplified) as follows

Theorem 5 *A ring R is unit regular if and only if for every element $a \in R$ there is a unit $u \in U(R)$ such that $aR \cap (a-u)R = 0$.*

Proof. First notice that if $e \in R$ is an idempotent in a ring R then $eR \cap aR \subseteq eaR$ for every $a \in R$, and, if $ea = 0$ then $eR \cap aR = 0$.

If R is unit regular, for every $a \in R$ there is a unit $u \in U(R)$ such that $a = au^{-1}a$. Thus $au^{-1}(a-u) = 0$ and since au^{-1} is an idempotent, $au^{-1}R \cap (a-u)R = 0$. Finally, since $aR = au^{-1}uR \subseteq au^{-1}R$ we obtain $aR \cap (a-u)R = 0$ (obviously $au^{-1}R \subseteq aR$ and so actually $au^{-1}R = aR$). Conversely, let $a \in R$ and $u \in U(R)$ with $aR \cap (a-u)R = 0$. Computing $au^{-1}(a-u) = (a-u+u)u^{-1}(a-u) = (a-u)u^{-1}(a-u) + a - u \in aR \cap (a-u)R$, we obtain $au^{-1}(a-u) = 0$ and hence $a = au^{-1}a$. ■

Remarks. 1) Another proof can be given using: *R is unit regular if and only if for every $a \in R$ there is a unit $u \in U(R)$ and an idempotent e such that $a = eu$.* Then $a - u = eu - u = (e-1)u$ and so $e(a-u) = 0$. Therefore, $eR \cap (a-u)R = 0$, and, since $aR = euR \subseteq eR$, this gives $aR \cap (a-u)R = 0$. Conversely, suppose that for an element $a \in R$ there is a unit $u \in U(R)$ such that $aR \cap (a-u)R = 0$. We consider $e = au^{-1}$ for which clearly $a = eu$ and we check e is an idempotent. Indeed, computing $e(a-u) = \underline{a}u^{-1}(a-u) = (a-u+u)u^{-1}(a-u) = \underline{(a-u)}u^{-1}(a-u) + \underline{a-u} \in aR \cap (a-u)R$, we obtain $e(a-u) = 0$. By right multiplication with u^{-1}, this finally gives $e(e-1) = 0$ or $e^2 = e$.

2) The (necessary and sufficient) conditions of Camillo-Khurana's characterization, can be reformulated as follows: *for every element $a \in R$ there is a unit $u \in U(R)$ such that $aR \cap (a-u)R = 0$,* **and $a - u$ is an idempotent**. According to our previous Theorem, the bold part is superfluous. Hence

Corollary 6 *Let R be a unit regular ring and $a \in R$ an arbitrary element. According to the previous Theorem, there are units $u \in U(R)$ such that $aR \cap (a-u)R = 0$. Among all these units, we can always choose a unit $w \in U(R)$ such that $a - w$ is an idempotent.*

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