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Research Article

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Idempotent matrices with invertible transpose

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Abstract: We prove that if the transpose of any 2×2 matrix over a division ring D, different from the identity matrix, is not invertible, then D is commutative.

Keywords: Idempotent matrix, invertible transpose, commutators, division ring

MSC 2010: 15B99, 16K99, 15A03, 15A09

1 Introduction

It has been well-known for a long time (since 1953) that the transpose of an invertible matrix over a division ring may not be invertible (see [2, p. 24, Exercise 3]). Gupta proved in [1] that the transpose of an invertible matrix over a division ring may be even nilpotent.

In this short note we show that, over a division ring, the transpose of an invertible matrix, different from the identity matrix, may be idempotent. Clearly, the existence of an idempotent matrix ($\neq I_2$) with invertible transpose is equivalent to the existence of an invertible matrix whose transpose is idempotent.

If $E = E^2$, then $E^t = (E^2)^t \neq (E^t)^2$ may happen, that is, the transpose of an idempotent matrix is not necessarily idempotent. As mentioned in [1], actually $(A^2)^t = (A^t)^2$ for every 2×2 matrix over a ring R is equivalent to the commutativity of R.

In closing, similarly to the results obtained in [1], we show that the nonexistence of such examples (except the identity matrix) implies the commutativity of the division ring.

2 The idempotent case

We start with a useful lemma.

Lemma 2.1. *A is a 2×2 idempotent matrix over a division ring D if and only if $A \in \{0_2, I_2\}$ or*

$$A = \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix} \quad or \quad A = \begin{bmatrix} 1 - yz & y \\ z - zyz & zy \end{bmatrix}$$

for some $c, y, z \in D$, $y \neq 0$.

Proof. Indeed, $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is idempotent if and only if $a^2 + bc = a$, $d^2 + cb = d$, $ab + bd = b$, $ca + dc = c$.

If $b \neq 0$, the third relation above gives $a = 1 - bdb^{-1}$. Denoting $b = y$ and $z = db^{-1}$, one gets $a = 1 - yz$, $d = zy$, and then $c = z - zyz$, so $A = \left[\begin{smallmatrix} 1-yz & y \\ z-zyz & zy \end{smallmatrix}\right]$, with $y \neq 0$.

If $b = 0$, then $a, d \in \{0, 1\}$, and, by analyzing each combination, one gets the rest of the idempotent matrices.

Conversely, for all matrices in the statement, one verifies $A^2 = A$. □

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Lemma 2.2. *If* $A = \left[\begin{smallmatrix} 1-yz & y \\ z(1-yz) & zy \end{smallmatrix}\right]$, *with* $y \neq 0 \neq z$, $y, z \in D$, *then* A^t *is not invertible if and only if* z *and* $(1 - yz)y^{-1}$ *commute.*

Proof. The transpose $A^t = \left[\begin{smallmatrix} 1-yz & z(1-yz) \\ y & zy \end{smallmatrix}\right]$ is not invertible if and only if its rows are (left) linearly dependent over D. This is equivalent to (left) linearly dependent $(1 - yz, z(1 - yz))$, $(1, y^{-1}zy)$ and so with $z(1 - yz) = (1 - yz)y^{-1}zy$. By right multiplication with y^{-1}, this is equivalent to commuting z and $(1 - yz)y^{-1}$. □

Example 2.3. As in [1], our example uses a division ring of quaternions over any field F in which $a^2 + b^2 + c^2 + d^2 = 0$ implies $a = 0$, $b = 0$, $c = 0$, $d = 0$ (for instance over $\mathbf{R}$ or $\mathbf{Q}$). It suffices to take (say) $y = i$ and $z = j$. Then, indeed, $z(1 - yz)y^{-1} = -1 + k \neq -1 - k = (1 - yz)y^{-1}z$, and so for

$$A = \begin{bmatrix} 1 - yz & y \\ z(1 - yz) & zy \end{bmatrix} = \begin{bmatrix} 1 - k & i \\ -i + j & -k \end{bmatrix},$$

we have

$$A^2 = A \quad \text{and} \quad (A^t)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -i - j \\ -j & -1 + k \end{bmatrix}.$$

In order to prove that the nonexistence of such examples (except I_2) implies the commutativity of the division ring, we first prove a result which is similar to the Straus' proof added in [1].

Proposition 2.4. *If* z, x *are two noncommuting elements in a division ring* D *such that the commutator* $[x, z] \neq 1$, *then there exists at most one element* y *in the coset* $x + C_z$ *such that* $(1 - yz)y^{-1} \in C_z$, *where* $C_z = \{c \in D : zc = cz\}$ *is the centralizer of* z *in* D.

Proof. Since z and x do not commute, obviously $z \neq 0 \neq x$. Suppose there are distinct $y, y' \in x + C_z$ such that both $(1 - yz)y^{-1}$, $(1 - y'z)(y')^{-1} \in C_z$, and denote $z_1 = (1 - yz)y^{-1}$ (this way $yz = 1 - z_1y$). Since y, y' belong to the same coset, $y' = y + c$, with $0 \neq c \in C_z$.

We compute

$$y'z = (c + y)z = zc + 1 - z_1y = 1 + (z + z_1)c - z_1(c + y) = 1 + (z + z_1)c - z_1y',$$

and so $1 - y'z = -(z + z_1)c + z_1y'$. Hence, by hypothesis, $(1 - y'z)(y')^{-1} = -(z + z_1)c(y')^{-1} + z_1 \in C_z$, and since $z, z_1, c \in C_z$ and $z + z_1 \neq 0 \neq c$, we get $(y')^{-1} \in C_z$. But then $y' \in C_z$, and so $x \in C_z$, a contradiction.

Note that $z + z_1 = 0$ amounts to $[y, z] = yz - zy = 1$ and, since $y \in x + C_z$, to $[x, z] = 1$. □

It is readily seen that the hypothesis on the commutator $[x, z] \neq 1$ is superfluous.

Lemma 2.5 (Bergman). (i) *If* $[x, z] = 1$, *then* $xz^n = z^nx + nz^{n-1}$.

(ii) *If all nonzero commutators are* $= 1$ *in a ring* R, *then* R *has characteristic* 2.

(iii) *If all nonzero commutators are* $= 1$ *in a ring* R *without zero divisors, then every noncentral element has square* 1.

(iv) *In a ring without zero divisors, there exist commutators which are neither* 0 *nor* 1.

Proof. (i) If $xz = zx + 1$, successive right multiplication by z and replacement of this relation give the stated equality.

(ii) Exchanging the roles of z and x, we get $-1 = 1$, so R has characteristic 2.

(iii) Take a noncentral element z, and some x with $[x, z] \neq 0$. Then, for $n = 3$ in (i) (using (ii)), we get $xz^3 = z^3x + z^2$, i.e., $z^2 = [x, z^3]$, so $z^2 = 1$ ($z^2 = 0$ is not possible, since $z \neq 0$ would be a zero divisor), i.e., every noncentral element has square 1.

(iv) Suppose that for any noncommuting elements z and x, we have $[x, z] = 1$. Then all the above holds, and since zx is not central (since $z \neq 0 \neq x$, it commutes with neither z or x), it should have square 1. However, $(zx)^2 = zxzx = z(zx + 1)x = z^2x^2 + zx = 1 + zx$ is a contradiction (R has no zero divisors). □

Theorem 2.6. *If* D *is a division ring such that the transpose of every idempotent matrix* $\neq I_2$ *over* D *is not invertible, then* D *is commutative.*

Proof. Suppose D is not commutative.

Since the transposes of 0_2 or $\left[\begin{smallmatrix} 0 & 0 \\ a & 1 \end{smallmatrix}\right]$ are not invertible, according to Lemma 2.1, it suffices to show that a matrix of the form $\left[\begin{smallmatrix} 1-yz & y \\ z(1-yz) & zy \end{smallmatrix}\right]$ has an invertible transpose. Then, using Lemma 2.2, it suffices to find two nonzero elements $y, z \in D$ such that $z(1-yz)y^{-1} \neq (1-yz)y^{-1}z$. Since $|C_z| \geq 2$, take a commutator $[x, z]$ which is neither 0 nor 1. Then (by Proposition 2.4) there exists at most one (nonzero) element $y \in x + C_z$ such that $z(1-yz)y^{-1} = (1-yz)y^{-1}z$. Since C_z contains 0 and 1, and both x and $x+1$ (which lie in $x + C_z$) are nonzero (the second one is nonzero since x is not central), it is possible to find $y \in \{x, x+1\}$ such that $z(1-yz)y^{-1} \neq (1-yz)y^{-1}z$. □

Corollary 2.7. *If D is a division ring such that the transpose of every idempotent 2×2 matrix over D is idempotent, then D is commutative.*

Corollary 2.8. *If the transpose of every 2×2 invertible matrix $\neq I_2$ over D is not idempotent, then D is commutative.*

As George Bergman noticed (private correspondence), Lemma 2.5 can be largely generalized.

Proposition 2.9. *In any noncommutative ring without zero divisors, not all commutators are central.*

Proof. The proof of Lemma 2.5 can be adapted. Denote by $Z(R)$ the center of R. The key ingredient is the following: if $0 \neq c \in Z(R)$, $a \in R$ and R has no zero divisors, then $ca \in Z(R)$ implies $a \in Z(R)$. □

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References

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