HYPERGROUPS ASSOCIATED WITH LATTICES

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Abstract. Hypergroups associated with modular lattices, respectively compactly generated lattices, are studied and characterized.

Introduction

The first hypergroup studied in this paper was introduced by Comer (in [1]) and furthermore studied by Konstantinidou M., Serafimidis K. and Mittas J. (see [5] and [6]).

If $(L, \vee, \wedge)$ is a lattice, then for every $(a, b) \in L^2$, set $a \circ b = \{x \in L \mid a \vee x = b \vee x = a \vee b\}$. We mention that a particular case of this hyperoperation had been already considered by Iacovlev B.V. (see [4]) and later by Scoppola C.M. (see [7]).

In what follows, we shall use notations and terminology from [2].

The following characterization (see [1], [5] and [6]) holds:

0. Theorem. *The lattice* $(L, \vee, \wedge)$ *is modular iff* $< L, \circ >$ *is a join space.*

In this paper, we shall give necessary and sufficient conditions in a join space $(H, \circ)$, such that H has a lattice structure $(H, \vee, \wedge)$, so that

$$a \circ b = \{x \in H | a \vee x = b \vee x = a \vee b\},$$

that is, so that the join space $(H, \circ)$ is associated with the (modular) lattice $(H, \vee, \wedge)$.

We also characterize the hypergroups associated with (modular) compactly generated lattices. Finally, the subhypergroups of hypergroups associated with lattices and the isomorphism of such hypergroups are analysed.

§1. First, we shall mention some important (for which follows) properties of a join space $< L, \circ >$ associated with a modular lattice $(L, \vee, \wedge)$.

One can verify these properties, by the equivalence $x \in a \circ a \Longleftrightarrow x \leq a$.

1. Proposition. *For a modular lattice L, the associated join space $< L, \circ >$ satisfies the following properties:*

(i) $\forall a \in L : a \in a \circ a$; $a \circ a$ *is a subhypergroup of* $< L, \circ >$;

(ii) $\forall (a,b) \in L^2 : \bigcup\limits_{\{a,b\} \subseteq x \circ x} x \circ x = (a \vee b) \circ (a \vee b)$;

(iii) $\forall (a,b) \in L^2 : a \circ a \cap b \circ b \subseteq (a \wedge b) \circ (a \wedge b)$ *and* $a \wedge b \in a \circ a \cap b \circ b$;

(iv) $\forall (a,b) \in L^2 : \{a,b\} \subseteq a \circ b \Longrightarrow a = b$;

(v) $\forall a \in L : a \circ a \circ a = a \circ a$;

(vi) $\forall (a,b) \in L^2 : a \circ b = [a \circ (a \vee b)] \cap [b \circ (a \vee b)]$;

(vii) *if* $a < b, a \circ b = \{b\} \cup \{x \in L | x < b, x || a, \nexists y \in L : a < y < b, x < y\}$, *where we denote by* $x || a$ *two incomparable elements of* L.

We notice that in a join space $< L, \circ >$, associated with a modular lattice, the following condition holds:

(α) $\forall (a,b) \in L^2, \exists x \in L, \exists t \in L : \{a,b\} \subseteq x \circ x, \bigcap\limits_{\{a,b\} \subseteq x \circ x} x \circ x = t \circ t, a \circ b = a \circ t \cap b \circ t.$

Moreover, if $a \in t \circ t - \{t\}$, then

$$a \circ t = \{t\} \cup \left\{ \begin{array}{c} u \in L | u \in t \circ t - \{t\}, u \notin a \circ a, a \notin u \circ u, \nexists y \in L : \\ a \in y \circ y - \{y\}, u \in y \circ y - \{y\}, y \in t \circ t - \{t\} \end{array} \right\}.$$

The condition (α) is equivalent to the set of conditions (ii), (vi) and (vii), written using only the hyperoperation "$\circ$" (not the order "$\leq$").

2. Theorem. *A join space $< H, \circ >$ is associated with a lattice $(H, \vee, \wedge)$ iff it satisfies (α) and the following conditions:*

(1) $\forall (a,b) \in H^2 : a/b = a \circ b$;

(2) $\forall a \in H : a \circ a \circ a = a \circ a$;

(3) $\forall (a,b) \in H^2, \exists s \in a \circ a \cap b \circ b : a \circ a \cap b \circ b \subseteq s \circ s$;

(4) $\forall (a,b) \in H^2 : \{a,b\} \subseteq a \circ b \Longleftrightarrow a = b$.

Proof. From Proposition 1, it follows that the above conditions are necessary. For the sufficiency, we define a binary relation on H as follows:

$$a \leq b \Longleftrightarrow a \in b \circ b \overset{(1)}{\Longleftrightarrow} b \in a \circ b.$$

This is an order on H according to (4) [reflexivity: $\forall a \in H : a \in a \circ a$], (2) [transitivity: $a \in b \circ b, b \in c \circ c \Longrightarrow a \in c \circ c \circ c \circ c = c \circ c \circ c = c \circ c$] and again (4) [antisymmetry].

In order to obtain a lattice structure, for arbitrary elements $a, b \in H$ we consider t, where

$$\bigcap_{\{a,b\} \subseteq x \circ x} x \circ x = t \circ t$$

and verify that $t = \sup(a, b)$. Indeed, $\{a, b\} \in t \circ t$ so that $a \leq t, b \leq t$; moreover, if $a \leq s, b \leq s$, then

$$t \in t \circ t = \bigcap_{\{a,b\} \subseteq x \circ x} x \circ x \subseteq s \circ s$$

because $\{a, b\} \in s \circ s$ so $t \leq s$. The antisymmetry proves that t is unique. On the other hand, for an arbitrary element $(a, b) \in H^2$, we consider s, such that

$$s \in a \circ a \cap b \circ b : a \circ a \cap b \circ b \subseteq s \circ s$$

and verify that $s = \inf(a, b)$. Obviously, $s \leq a, s \leq b$ and if $u \leq a, u \leq b$ then $u \in a \circ a \cap b \circ b \subseteq s \circ s$, so $u \leq s$. The element s is unique because $\{s_1, s_2\} \subseteq a \circ a \cap b \circ b \subseteq s_1 \circ s_1 \cap s_2 \circ s_2$ implies $s_1 \in s_2 \circ s_2$ and $s_2 \in s_1 \circ s_1$ and so $s_1 = s_2$. Hence, $(H, \sup, \inf)$ is a lattice. Its modularity is easily checked.

In what follows, we use the standard notations

$$\sup(a, b) = a \vee b, \ \inf(a, b) = a \wedge b.$$

Now, we verify the inclusion $a \circ b \subseteq \{x \in H | a \vee x = b \vee x = a \vee b\}$: let $x \in a \circ b$; from $\{a, b\} \subseteq t \circ t$ where $t = a \vee b$ it results $x \in a \circ b \subseteq t \circ t \circ t \circ t = t \circ t$ and so $x \in (a \vee b) \circ (a \vee b)$ that is $x \leq a \vee b$. Hence $a \vee x \leq a \vee b$ and $b \vee x \leq a \vee b$. But $\{b, x\} \subseteq (b \vee x) \circ (b \vee x)$ and so $b \circ x \subseteq (b \vee x) \circ (b \vee x) \circ (b \vee x) \circ (b \vee x) = (b \vee x) \circ (b \vee x)$. Using (1), we have $x \in a \circ b = a/b$ and so $a \in b \circ x \subseteq (b \vee x) \circ (b \vee x)$ whence $a \leq b \vee x$ so $a \vee b \leq b \vee x$. We obtain $a \vee b = b \vee x$. Similarly, we have $a \vee b = a \vee x$ and hence $x \in \{x \in H | a \vee x = b \vee x = a \vee b\}$.

Conversely, take $x \in H$ such that $a \vee x = b \vee x = a \vee b$.

It results $x \leq a \vee x = b \vee x = a \vee b$.

We distinguish the following cases:

Case 1: if $a = b$ then $x \leq a$ and so $x \in a \circ a = a \circ b$.

Case 2: if $b < a$ then $x \leq a = a \vee b$. If $x = a$ nothing is to be proved. If $x < a$ then $b \leq x$ is not possible (otherwise $a = b \vee x = x$) nor $b \geq x$ (otherwise $a = b \vee x = b$) and so $b \| x$. Moreover, there is no element $y \in H$ such that $b < y < a, x < y$ (otherwise $a = b \vee x \leq y < a$). Therefore $x \in a \circ b$ using (α).

Case 3: if $a < b$ one verifies similarly that $x \in a \circ b$.

Case 4: if $a||b$ then $a < a \vee b, b < a \vee b$. We first check that $x \in a \circ (a \vee b)$. This is clear for $x = a \vee b$ so in what follows we suppose $x \neq a \vee b$. Now $x \leq a \Longrightarrow a \vee x \leq a \Longleftrightarrow a \vee b \leq a \Longleftrightarrow a \vee b = a$ and analogously $a \leq x \Longrightarrow x = a \vee b$, both contradictions so that $x||a$ and $x < a \vee b$. As above $x \in \{a \vee b\} \cup \{u \in H | u < a \vee b, u||a, \nexists y \in H : a < y < a \vee b, u < y\} = = a \circ (a \vee b)$.

Similarly, $x \in b \circ (a \vee b)$ and so $x \in a \circ (a \vee b) \cap b \circ (a \vee b) = a \circ t \cap b \circ t$. Hence, using the condition (α), $x \in a \circ b$ and this completes our proof. ∎

3. Remark.

1°. If $(H, \circ)$ has an identity, then the corresponding lattice has this element as zero. Indeed, $\forall a \in H : a \in a \circ e \Leftrightarrow e \leq a$.

2°. If it is considered in a join space only the first part of the condition (α) (i.e. $\forall a, b \in H, \exists x \in H, \exists t \in H : \{a, b\} \subseteq x \circ x, \bigcup_{\{a,b\} \subseteq x \circ x} x \circ x = t \circ t$), then in general an enlargement of the initial hypergroup is obtained (i.e. $a \circ b \underset{\neq}{\subseteq} \{x \in H | a \vee x = b \vee x = a \vee b\}$ could hold).

§2. We shall consider now the case of hypergroups associated with (modular) compactly generated lattices.

First, recall that in a lattice L an element c is called *compact* if for each subset $X \subseteq L$, and $c \leq \bigvee_{x \in X} x \overset{\text{not}}{=} \bigvee X$ there is a finite subset $F \subseteq X$ such that $c \leq \bigvee F$. A complete lattice is called *compactly generated* (*algebraic*) if each element is a join of compact elements.

Now, we shall mention some properties which hold in a quasihypergroup associated with a complete lattice. We shall use the notations:

$$\text{for } x \in L,\ I_p(x) = \{e \in L \mid x \in x \circ e \cup e \circ x\} \text{ and}$$
$$\text{for } A \subseteq L,\ I_p(A) = \bigcup_{x \in A} I_p(x).$$

One can verify these properties, by the equivalence:

$$x \in I_p(y) \Longleftrightarrow x \leq y.$$

4. Proposition. *Let* $(L, \vee, \wedge)$ *be a complete lattice and* $(H, \circ)$ *the associated quasihypergroup. The following properties hold:*

(1) $\forall x \in L : x = \bigvee I_p(x)$;

(2) *if* $x = \bigvee U$ *then* $\forall u \in U, u \in I_p(x)$;

(3) $\forall x \in L, I_p(x) = x \circ x$;

(4) *for each* $x, y \in L$ *we have* $x \leq y \Leftrightarrow x \circ x \subseteq y \circ y \Leftrightarrow x \in y \circ y$;

(5) *for each* $X \subseteq L$, $\bigcup_{F \subseteq X, F \text{ finite}} I_p(\bigvee F) \subseteq I_p(\bigvee X)$;

(6) $I_p(x) = I_p(y) \Leftrightarrow x = y$;

(7) $\forall u \in L$ *if* $U = \{x \in L | x \leq u\}$ *then* $I_p(u) = I_p(U)$;

(8) $I_p(X) = I_p(Y) \Longrightarrow \bigvee X = \bigvee Y$, *but the converse is false.*

We shall establish necessary and sufficient conditions in a join space $< L, \circ >$, such that $< L, \circ >$ can be associated with a (modular) compactly generated lattice:

5. Theorem. *Let* $< L, \circ >$ *be a join space, which can be associated with a modula lattice* $(L, \vee, \wedge)$, *that is* $(L, \circ)$ *satisfies the conditions of Theorem* 2. *The lattice* $(L, \vee, \wedge)$ *is a compactly generated one iff the join space* $< L, \circ >$ *satisfies the conditions of Theorem* 2 *and furthermore the following ones:*

1°. $\forall X \subseteq L,\ X \neq \emptyset,\ \exists y \in L,\ \exists a_X \in L : X \subseteq y \circ y$ *and* $\bigcap_{X \subseteq y \circ y} y \circ y = a_X \circ a_X$.

2°. $\exists T \subseteq L,\ T \neq \emptyset$, *such that* $\forall X \subseteq L$ *the following assertion holds:*
$\forall t \in T$, *if* $t \in a_X \circ a_X$, *then* $\exists F \subseteq X$, *F finite, such that* $t \in a_F \circ a_F$.

3°. $\forall x \in L,\ \exists S \subseteq T \cap x \circ x,\ S \neq \emptyset$, *such that* $S \subseteq c \circ c \Longrightarrow x \in c \circ c$.

Proof. By Theorem 2, we can define on L the following order:

$$x \leq y \Longleftrightarrow x \in y \circ y$$

and we obtain a lattice modular $(L, \leq)$, using the conditions of Theorem 2.

The condition 1° asserts that for any non-empty subset X of L, there is $\sup X \in L$.

Indeed, if 1° is satisfied, then $\forall x \in X,\ x \in a_X \circ a_X$, that means $x \leq a_X$ and if $b \in X$ such that $x \leq b, \forall x \in X$, then $a_X \in a_X \circ a_X = \bigcap_{X \subseteq y \circ y} y \circ y \subset b \circ b$, since $X \subseteq b \circ b$; so, $a_X \leq b$. Therefore, $\exists \sup X$ and $\sup X = a_X$.

Conversely, if $\forall X \subseteq L, X \neq \emptyset$, there is $\sup X = a_X \in L$, then $X \subseteq a_X \circ a_X$ and for any $b \in L$ such that $x \leq b, \forall x \in X$, we have $a_X \leq b$, that means if $b \in L$, such that $X \subseteq b \circ b$, then $a_X \in b \circ b$. Hence $a_X \circ a_X \subseteq (b \circ b) \circ (b \circ b) = b \circ b$, by condition (2) of Theorem 2, whence $a_X \circ a_X \subseteq \bigcap_{X \subseteq b \circ b} b \circ b \subseteq a_X \circ a_X$, so $\bigcap_{X \subseteq b \circ b} b \circ b = a_X \circ a_X$. The condition 2° says that every $t \in T$ is a compact element. Indeed, it results by the equivalences:

$$t \in a_X \circ a_X \Longleftrightarrow t \leq a_X \Longleftrightarrow t \leq \sup X.$$

Finally, by the third condition it results that $\forall x \in X$, $\exists S \subseteq T$, $x = a_S$, that means x is a join of compact elements.

Therefore, the conditions 1°, 2° and 3° show that the lattice L is compactly generated. ■

In the following, we shall characterize the compactly generated lattices.

In order to do it, we shall study the subhypergroups of join spaces associated with lattices and isomorphisms of such join spaces are established.

6. Proposition. *Let L be a modular lattice. A subset I of L is an (invertible) subhypergroup of $< L, \circ >$ iff I is an ideal of L.*

Proof. If I is a subhypergroup of $< L, \circ >$, for every $(a, b) \in I^2$, we have $a \vee b \in a \circ b \subseteq I$. Moreover, if for $a \in I$ one takes $x \leq a$, $x \in L$ then $x \in a \circ a \subseteq I$ and so, I is an ideal of L.

Conversely, let I be an ideal of L. For $(a, b) \in I^2$, if $t \in a \circ b$ then $t \leq a \vee b$ and so $t \in I$ (I being ideal). For every $(a, b) \in I^2$ there is an element $x = a \vee b \in I$ such that $a \in b \circ x$. Hence I is a subhypergroup of $< L, \circ >$.

We finally remark that any multiplicatively closed part of $< L, \circ >$ is a subhypergroup, which is also invertible. ■

7. Proposition. *If L is a modular lattice, then $\omega_{<L,\circ>} = L$.*

Proof. It is sufficient to verify that the only ultraclosed subhypergroup of $< L, \circ >$ is $< L, \circ >$ itself.

Suppose that I is an ultraclosed subhypergroup of L. If $I \neq L$, take $a \notin I$ and $t \in I$. Then $a \vee t \notin I$ and so $a \vee t \in (a \circ I) \cap (a \circ (L - I))$. Hence $(a \circ I) \cap (a \circ (L - I)) = \emptyset$ holds for every $a \in L$ only if $I = L$. ■

Let us consider now a dual hyperoperation, i.e.

$$\forall (a, b) \in L^2,\ a * b = \{x \in L \mid a \wedge x = b \wedge x = a \wedge b\}.$$

8. Proposition. *For a lattice L let $f : L \to L$ be a bijective map. The following conditions are equivalent:*

(1) $f(a \vee b) = f(a) \wedge f(b)$, $\forall (a, b) \in L^2$,

(2) $f(a \circ b) = f(a) * f(b)$, $\forall (a, b) \in L^2$.

Proof. (1) $\Longrightarrow$ (2) Clearly $f(a \circ b) = \{f(x) | x \in L, x \vee a = x \vee b = a \vee b\}$ and so $f(x) \in f(a) * f(b)$ by (1). Conversely, if $t \in f(a) * f(b)$, using the surjectivity of f, there is an element $x \in L$ such that $t = f(x)$ and so $f(x \vee a) = f(x \vee b) = f(a \vee b)$, again by (1). Hence, owing to the injectivity of f, $x \in a \circ b$ and $t \in f(a \circ b)$.

(2) $\Longrightarrow$ (1) For every $x \in a \circ b$ our hypothesis gives $f(x) \in f(a) * f(b)$ and so $f(x) \wedge f(a) = f(x) \wedge f(b) = f(a) \wedge f(b) \leq f(x)$. Taking $x = a \vee b$ one obtains

$f(a) \wedge f(b) \leq f(a \vee b)$. Conversely, observe that $f(x) \in f(a) * f(a)$ holds for each $x \in a \circ a$ (and each $a \in L$). Hence $f(a) = f(x) \wedge f(a)$ and $f(a) \leq f(x)$. Therefore, if $x \leq a$, then $f(a) \leq f(x)$, whence it results $f(a \vee b) \leq f(a) \wedge f(b)$. ∎

By a similar way as in the former Proposition, it is possible to provee the following results:

(I) Let $(L, \vee, \wedge)$ be a lattice and $f : L \to L$ be a bijective map. The following conditions are equivalent:

(1') $f(a \wedge b) = f(a) \vee f(b),\ \forall (a, b) \in L^2$,

(2') $f(a * b) = f(a) \circ f(b),\ \forall (a, b) \in L^2$.

(II) Let $(L, \vee, \wedge)$ and $(L', \vee, \wedge)$ be modular lattices, $f : L \to L'$ be a function and $(L, \circ)$ resp. $(L', \circ)$ the associated hypergroups. Then the following conditions are equivalent:

(3) f is bijective and for any $(x, y) \in L^2$, $f(x \vee y) = f(x) \vee f(y)$;

(4) f is a hypergroup isomorphism (i.e. f bijection and $\forall\, (x, y) \in L^2$, $f(x \circ y) = f(x) \circ f(y)$).

(III) Let $(L, \vee, \wedge)$ and $(L', \vee, \wedge)$ be modular lattices, $f : L \to L'$ be a function and $(L, *)$, resp. $(L', *)$ the associated hypergroups. Then the following conditions are equivalent:

(3') f is bijective and for any $(x, y) \in L^2$, $f(x \wedge y) = f(x) \wedge f(y)$;

(4') f is a hypergroup isomorphism (i.e. f bijection and $\forall\, (x, y) \in L^2$, $f(x * y) = f(x) * f(y)$).

9. Example. In a Boole lattice L, if $f : L \to L$ is defined by $f(a) = a'$, $\forall\, a \in L$, then the conditions (1) and (1') are satisfied, so also (2) and (2') are satisfied.

10. Remarks.

1. The lattice $(L, \vee, \wedge)$ is a modular one iff $< L, * >$ is a join space. Moreover, if L is a modular lattice, then the following equivalences hold:

   $(L, *)$ is a regular hypergroup $\iff$ $(L, *)$ is a regular reversible hypergroup $\iff$ $(L, *)$ is a canonical hypergroup $\iff$ L has a greatest element.

   The proof of this result is similar to that one concerning the join space $< L, \circ >$ (see [1], [5] and [6]).

2. In a modular lattice $(L, \vee, \wedge)$, a subset I is an (invertible) subhypergroup of $< L, * >$ iff I is a filter of L. Moreover, $\omega_{<L,*>} = L$.

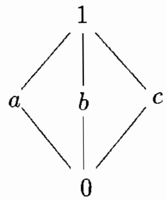
   The proof is similar to the proof of Proposition 6.

We mention that the above Proposition and the similar results can be used in order to verify that some lattice provide, together with the above hyperoperations, hypergroups.

Indeed, if such a bijection $f : L \longrightarrow L$ exists, where L is a lattice, such that for instance $\forall (x,y) \in L^2$, $f(x \vee y) = f(x) \vee f(y)$, then one can deduce the associativity of "$*$" from the associativity of "$\circ$" in the following way:

$$(x_1 * y_1) * z_1 = (f(x) * f(y)) * f(z) = f(x \circ y) * f(z) = f((x \circ y) \circ z) =$$
$$= f(x \circ (y \circ z)) = f(x) * f(y \circ z) = f(x) * (f(y) * f(z)) = x_1 * (y_1 * z_1).$$

For instance, for the non–distributive 5-element lattice M_5



one can use the bijective map $f = \begin{pmatrix} 0 & a & b & c & 1 \\ 1 & a & b & c & 0 \end{pmatrix}$.

Indeed, $f(a \vee b) = f(1) = 0 = a \wedge b = f(a) \wedge f(b)$, $f(a \vee 1) = f(1) = = 0 = a \wedge 0 = f(a) \wedge f(1)$, $f(a \vee 0) = f(a) = a = a \wedge 1 = f(a) \wedge f(0)$ and $f(0 \vee 1) = f(1) = 0 = 1 \wedge 0 = f(0) \wedge f(1)$.

Moreover, in this case, we obtain that $(M_5, \circ)$ and $(M_5, *)$ are isomorphic.

Finally, we present the following characterization theorem for a modular compactly generated lattice:

11. Theorem. *A modular lattice $(L, \vee, \wedge)$ is compactly generated iff there is a join–semilattice with zero L' such that $(L, \circ)$ is isomorphic to $(S(L'), \circ)$ and $(L, *)$ is isomorphic with $(S(L'), *)$, where $S(L')$ denotes the set of all the multiplicatively closed parts of $(L', \circ)$.*

Proof. By (II) and (III) it results that $[(L, \circ) \simeq (S(L'), \circ)$ and $(L, *) \simeq \simeq (S(L'), *)]$ iff the lattices L and $S(L')$ are isomorphic.

On the other hand, by the proof of Proposition 6 it results that any multiplicatively closed part of $< L', \circ >$ is a subhypergroup and a subset of L' is a subhypergroup of $< L', \circ >$ iff it is an ideal of the lattice $(L', \vee, \wedge)$.

Finally, to complete the proof of this Theorem, one uses Th.13, section 2, Ch.II, [3], which says that a lattice is compactly generated iff it is isomorphic to the lattice of all ideals of a join–sublattice (i.e. a partially ordered set, for which $\sup\{a, b\}$ exists for any two elements a and b) with 0. ■

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Accepted: 28.04.1999