An application of solving Diophantine quadratic equations

Grigore Călugăreanu

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Abstract

We find the positive integer solutions for the equations $\frac{1}{x}+\frac{1}{y}=1$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

The Diophantine quadratic equations below can be solved using suitable software (e.g. [1]). The ambitious reader will not use this and find an independent proof.

1 The equation $\frac{1}{x}+\frac{1}{y}=1$ for positive integers

The equation is equivalent to $xy=x+y$ which has only two solutions:

$(0,0),(2,2)$. Hence for positive integers we have only one solution

$$\frac{1}{2}+\frac{1}{2}=1.$$

2 The equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ for positive integers

For $n\geq 1$ we consider $\frac{1}{n}+\frac{1}{x}+\frac{1}{y}=1$, that is, $(n-1)xy-nx-ny=0$.

$n=1$: the equation becomes $x+y=0$ without positive integer solutions.

$n=2$: the equation $xy-2x-2y=0$ has six integer solutions:

$(0,0),(-2,1),(1,-2),(4,4),(3,6),(6,3)$. Hence for positive integers, we have only

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1,$$

and permutations.

$n=3$: the equation $2xy-3x-3y=0$ has six solutions:

$(0,0),(-3,1),(1,-3),(3,3),(2,6),(6,2)$. Hence for positive integers, we have only

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1}{3}+\frac{1}{2}+\frac{1}{6}=1$$

and permutations (the second solution is already a permutation of the $n=2$ case).

$n=4$: the equation $3xy-4x-4y=0$ has five solutions:

$(0,0),(-4,1),(1,-4),(2,4),(4,2)$. Hence for positive integers, we have only

$$\frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1$$

already a permutation of the $n=2$ case.

$n=5$: the equation $4xy-5x-5y=0$ has three solutions:

$(0,0),(-5,1),(1,-5)$. Hence for positive integers, we have no solutions.

$n=6$: the equation $5xy-6x-6y=0$ has five solutions:

$(0,0),(-6,1),(1,-6),(2,3),(2,2)$. Hence for positive integers, we have only

$$\frac{1}{6}+\frac{1}{2}+\frac{1}{3}=1$$

already a permutation of the $n=2$ case.

For $n>6$ the equation $(n-1)xy-nx-ny=0$ has (only) three solutions:

$(0,0),(-n,1),(1,-n)$.

Indeed, if $x\neq 1\neq y$, we can prove that $(n-1)xy>n(x+y)$. Hence for positive integers, we have no solutions.

Summarizing, up to permutations, the equation has only three solutions:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1.$$

Who wants to continue with $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=1$?

References

[1] Matthews http://www.numbertheory.org/php/generalquadratic.php