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PURITY IN Γ-LATTICES*

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1. Introduction

Let $(\Gamma, \cdot, 1)$ be a monoid. A lattice L is called **Γ-latice** if it is provided with a multiplication $\varphi : \Gamma \times L \to L$ (we shall denote by $\gamma a = \varphi(\gamma, a)$) which satisfies the following axioms

$\Gamma 1 : \gamma a \leq a$
$\Gamma 2 : \gamma(a \vee b) = \gamma a \vee \gamma b$
$\Gamma 3 : (\gamma\gamma')a = \gamma(\gamma' a)$
$\Gamma 4 : 1.a = a$.

The source of this enrichment of the notion of lattice is (see [3]) the lattice of all the submodules of a given module M over a commutative ring R with identity. Indeed, this is an algebraic modular lattice such that the commutative monoid of the principal ideals of R operates on the submodules in a natural way $\varphi(rR, A) = rA$ $(r \in R, A \leq M)$.

Remark 1.1. *Actually, this monoid naturally acts also on the quotient modules.*

As in [1] we use the quotient sublattice notation $b/a = \{c \in L | a \leq c \leq b\}$. All the lattices have 0 and 1.

2. Elementary results

In what follows Γ will denote a (non-necessary commutative) monoid.

LEMMA 2.1 *In any Γ-lattice,* $\gamma.0 = 0, \forall \gamma \in \Gamma$.

Indeed, from Γ1 we derive $\gamma.0 \leq 0$, and hence $\gamma.0 = 0, \forall\gamma \in \Gamma$. □

– One can consider, for a fixed $\gamma \in \Gamma$, the function $\varphi_\gamma : L \to L, \varphi_\gamma(a) = \gamma a, \forall a \in L$. The axiom Γ2 implies that φ_γ is an upper semi-morphism. Hence

LEMMA 2.2 *φ_γ is an order-preserving morphism.*

Indeed, $a \leq b \Rightarrow a \vee b = b \Rightarrow \varphi_\gamma(a \vee b) = \varphi_\gamma(b) \Rightarrow \varphi_\gamma(a) \vee \varphi_\gamma(b) = \varphi_\gamma(b) \Rightarrow \varphi_\gamma(a) \leq \varphi_\gamma(b)$. □

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Hence

LEMMA 2.3 (i) $a \leq b \Rightarrow \gamma a \leq \gamma b$. *Moreover,*
(ii) $\gamma(a \wedge b) \ \leq \ \gamma a \wedge \gamma b$.

Indeed, from (i) we get $a \wedge b \leq a \Rightarrow \gamma(a \wedge b) \leq \gamma a$ and similarly $\gamma(a \wedge b) \leq$ $\leq \gamma b$; hence $\gamma(a \wedge b) \ \leq \ \gamma a \wedge \gamma b$. □

A subset B of a Γ-lattice L is called Γ-**stable** if $\forall \gamma \in \Gamma, \gamma B \subseteq B$.

It is clear (using $\Gamma 1$) that the sublattices $a/0$ are Γ-stable and that in general not every sublattice $1/a$ (or b/a) is Γ-stable.

In a Γ-lattice an element $d \in L$ is called *c*-**divisible** if $\forall \gamma \in \Gamma : \gamma d = d$.

PROPOSITION 2.1 *A sublattice b/a is a Γ-stable iff a is c-divisible.*

Proof. If b/a is Γ-stable then $a \leq \gamma a$ and $\Gamma 1$ completes the equality. Conversely, from 2.3 we derive $a = \gamma a \leq \gamma c \leq c \leq b$ for each $c \in b/a$. □

One should define the notion of Γ-**sublattice** as a Γ-stable sublattice because in this way each Γ-sublattice has (by restriction) a natural structure of Γ-lattice.

Reconsidering the remark we have made in the introduction (concerning the natural action of the monoid on quotient modules) we shall consider on quotient sublattices b/a the following Γ-lattice structure:
$\forall \gamma \in \Gamma, \forall c \in b/a : \gamma * c = (\gamma c) \vee a$,
enlarging in this way the notion of Γ-sublattice.

Remark 2.1 *Surely, if a is c-divisible this is the natural Γ-lattice structure on b/a obtained by restriction.*

3. *c*-Pure elements

In what follows we shall use the following definition: in a Γ-lattice L an element p is *c*-**pure** if $\gamma p = p \wedge \gamma 1, \forall \gamma \in \Gamma$.

Remark 3.1 $\gamma p \leq p \wedge \gamma 1, \forall \gamma \in \Gamma, \forall p \in L$ *in every* Γ*-lattice* L.

Indeed, $\Gamma 1 \Rightarrow \gamma p \leq p$ and from the previous lemma (i) $p \leq 1 \Rightarrow \gamma p \leq \gamma 1$. □

PROPOSITION 3.1 *If L is a modular Γ-lattice and $p \in L$ has a complement in L then p is c-pure.*

Proof. Let a be a complement of p in L. Then $(\gamma 1) \wedge p = \gamma(p \vee a) \wedge p \overset{\Gamma 2}{=}$ $\overset{\Gamma 2}{=} ((\gamma p) \vee (\gamma a)) \wedge p \overset{\Gamma 1 + mod}{=} (\gamma p) \vee ((\gamma a) \wedge p) = (\gamma p) \vee 0 = \gamma p$ holds because $\gamma a \wedge p \overset{\Gamma 1}{\leq} a \wedge p = 0$. □

PROPOSITION 3.2 *In any Γ-lattice the c-purity is a transitive property.*

Proof. If L is a Γ-lattice, and $b \in L$ then (by $\Gamma 1$ and (i)) the sublattice $b/0$ is a Γ-lattice too.

Now, if $a \leq b$, a is c-pure in $b/0$ and b is c-pure in L we immediately derive $\gamma a = a \wedge \gamma b = a \wedge (b \wedge (\gamma 1)) = (a \wedge b) \wedge \gamma 1 = a \wedge \gamma 1$. □

PROPOSITION 3.3 *Let $a \leq b \leq c$ in a Γ-lattice L. If a is c-pure in $c/0$ then a is c-pure in $b/0$ too.*

Proof. Again, together with L, $b/0$ and $c/0$ are Γ-lattices too. The following computation gives the proof: $\gamma a = a \wedge \gamma c = a \wedge \gamma(b \vee c) \overset{\Gamma 2}{=} a \wedge ((\gamma b) \vee (\gamma c)) = a \wedge \gamma b$ (one uses (i) for $\gamma b \leq \gamma c$). □

LEMMA 3.1 *Each c-divisible element is also c-pure.*

Proof. Indeed, for each $\gamma \in \Gamma$ we have $a = \gamma a \leq \gamma 1$ and hence $a \wedge \gamma 1 = a = \gamma a$. □

The extension given in the previous section for the notion of Γ-sublattice permits us to define **relative** c-purity and to prove also other properties.

PROPOSITION 3.4 *Let $a \leq b$ be elements in a modular Γ-lattice L. If b is c-pure in L then b is also c-pure in $1/a$.*

Proof. First of all, b is c-pure in $1/a$ iff $\gamma * b = b \wedge (\gamma * 1), \forall \gamma \in \Gamma$. This is equivalent to $(\gamma b) \vee a = b \wedge ((\gamma 1) \vee a)$ and, by modularity, to $(\gamma b) \vee a = (b \wedge (\gamma 1)) \vee a$, true if b is c-pure in L. □

PROPOSITION 3.5 *Let $a \leq b$ be elements in a modular Γ-lattice L. If a is c-pure in L and b is c-pure in $1/a$ then b is c-pure in L.*

Proof. First, as above, b is c-pure in $1/a$ iff $\gamma * b = b \wedge (\gamma * 1), \forall \gamma \in \Gamma$ iff $(\gamma b) \vee a = (b \wedge (\gamma 1)) \vee a$.

Next, $(\gamma b) \wedge a = \gamma a = a \wedge (\gamma 1) = (b \wedge (\gamma 1)) \wedge a$ (the first equality: $\gamma a \leq \gamma b, \gamma a \leq a \Rightarrow \gamma a \leq (\gamma b) \wedge a$ and $(\gamma b) \wedge a \leq (\gamma 1) \wedge a = \gamma a$). So, again by modularity, $\gamma b = b \wedge \gamma 1$ follows from $\gamma b \leq b \wedge \gamma 1$ (see Remark 3.1). □

LEMMA 3.2 *Let L be an algebraic Γ-lattice and $a \in L$. If for each compact element $c \leq a$ there is a pure element $b \in L$ such that $c \leq b \leq a$ then a is pure.*

Proof. The lattice L being algebraic (compactly generated), according to 3.1 it suffices to prove that for each compact element c of L, and for each $\gamma \in \Gamma$, $c \leq a \wedge \gamma 1$ implies $c \leq \gamma a$.

Indeed, from $c \leq a \wedge \gamma 1 \leq a$ we have a pure element b in L such that $c \leq b \leq a$. But $c \leq \gamma 1$, $c \leq b$ imply $c \leq b \wedge \gamma 1 = \gamma b \leq \gamma a$ (using also 2.3). □

PROPOSITION 3.6 *In an algebraic Γ-lattice L a union of c-pure elements is c-pure.*

Proof. One uses the above lemma. See [2]. □

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